Scattering structures of linear Vlasov equations

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Linearized Vlasov-Poisson equations

- Non linear Vlasov-Poisson equations
with non homogeneous background ion density $\rho_{\rm ref}$

$$\left\{\begin{array}{lll} \partial_t f + v\partial_x f - E\partial_v f = 0, & t>0, & (x,v)\in I\times\mathbb{R}, \\ \partial_t E = 1^* \int_{\mathbb{R}} vfdv, & t>0, & x\in I, \\ \partial_x E = \rho_{\rm ref}(x) - \int_{\mathbb{R}} fdv, & t>0, & x\in I. \end{array}\right.$$

- Linearize around $f(t,x,v) = f_0(x,v) + g(t,x,v)$ and $E(t,x) = E_0(x) + F(t,x)$.

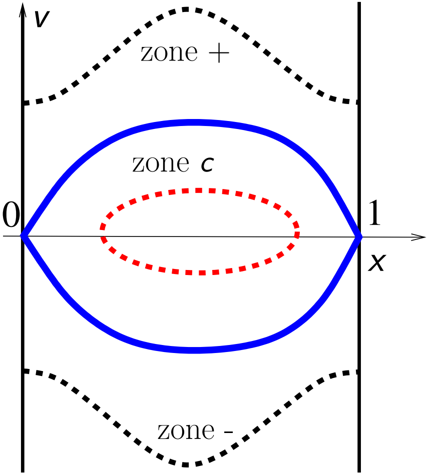
- It yields **model 1+1 linearized Vlasov-Poisson-Ampère system**

$$\left\{\begin{array}{lll} \partial_t g + v\partial_x g - E_0\partial_v g - F\partial_v f_0 = 0, & t>0, & (x,v)\in I\times\mathbb{R}, \\ \partial_t F = 1^* \int_{\mathbb{R}} vgdv, & t>0, & x\in I, \\ \partial_x F = -\int_{\mathbb{R}} gdv, & t>0, & x\in I, \end{array}\right. \quad (1)$$

equipped with periodic boundary conditions.

- Dorman, Landau damping of resonance oscillations in inhomogeneous plasmas, 70'.
- Campa and Chavanis, Dynamical stability criterion for inhomogeneous quasi-stationary states in long-range systems, 2010
- Barré, Olivetti and Yamaguchi, Landau damping and inhomogeneous reference states, 2015.
- Hutchinson, Electron holes in phase space : What they are and why they matter, 2017

Electron hole



The central zone of trapped particules : electron hole.
Zone + is above the separatrix. Zone - is below the separatrix.

Quadratic energy estimates

Set $M(x,v)=\sqrt{f_0(x,v)}=\sqrt{n_0(x)G(v)}=\exp\left(-\frac{v^2}{4}+\frac{\varphi_0(x)}{2}\right)$ and the function $u=\frac{g}{M}$

$$\left\{\begin{array}{lll} \partial_t u + v\partial_x u - E_0\partial_v u & = -vMF, & t>0, \quad (x,v)\in I\times\mathbb{R}, \\ \partial_t F & = 1^*\int_{\mathbb{R}} uvMdv, & t>0, \quad x\in I, \end{array}\right.$$

One has the energy identity

$$\frac{d}{dt}\left(\int_I\int_{\mathbb{R}} u^2 dvdx + \int_I F^2 dx\right) = 0$$

Define $U=(u,F)\in X := L^2(I\times\mathbb{R})\times L_0^2(I)$. One has

$$U'(t)=\mathbf{i}HU(t), \qquad H^*=H,$$

where $H=H_0+K$ (K compact perturbation in v but non compact in x)

$$\mathbf{i}H_0=\left(\begin{array}{c|c} -v\partial_x+E_0\partial_v & 0 \\ \hline 0 & 0 \end{array}\right) \text{ and } \mathbf{i}K=\left(\begin{array}{c|c} 0 & -vM \\ \hline 1^*\int_v vM & 0 \end{array}\right)$$

The Gauss law

$$\left\{(u,F)\in X \mid \int_{\mathbb{R}} uMdv+\partial_x F=0\right\}\subset \ker H.$$

Hiding the electric field : a technical tool

- Make the change of unknown

$$w(x,v,t) = u(x,v,t) + \gamma(x)M(x,v)F(x,t)$$

where the weight $\gamma(x)$ satisfies the Ricatti equation

$$\partial_x \gamma + \alpha^2\gamma^2 \exp\varphi_0 = 1, \qquad \alpha = (2\pi)^{\frac{1}{4}}, \qquad x \in \mathbb{T} = [0,1]_{\mathrm{per}}.$$

Then the total energy is preserved : $\int_{\mathbb{T}\times\mathbb{R}} w^2 dvdx = \int_{\mathbb{T}\times\mathbb{R}} u^2 dvdx + \int_{\mathbb{T}} F^2$.

- Moreover, w satisfies an autonomous equation **without Gauss constraint**

$$\partial_t w = i\mathcal{H}w, \quad \mathcal{H} = i\mathcal{H}_0 + i\mathcal{K},$$

where $i\mathcal{H}_0 = -\left(v\partial_x w - E_0(x)\partial_v\right)$ is the free transport operator, and

$$i\mathcal{K}w = \gamma\left(vM\int_{\mathbb{R}} wMdv - M\int_{\mathbb{R}} wvMdv\right) + \gamma M\int_{\mathbb{T}\times\mathbb{R}} wvMdv - \left(\int_{\mathbb{T}\times\mathbb{R}} w\gamma M\right) Mv.$$

Scattering in a Nutshell

X Hilbert space, two closed self-adjoint operators $H_0^* = H_0$ and $K^* = K$

$$U'(t) = iHU(t) \text{ with } H = H_0 + K.$$

Typically H_0 is unbounded but "simple" and K is "small".

- Goal : compare $U'(t) = e^{iHt}U_0$ with $\widetilde{U}'(t) = e^{iH_0 t}\widetilde{U}_0$.
- Scattering correspond to U_0 in the continuous spectrum of H. Then

$$\lim_{t\to\pm\infty} (U(t), V) = 0 \quad \forall V \in X.$$

Among the possible tools

a) explicit calculation of the spectrum (Rege's poster, $\omega_c \neq 0$)

b) the Lipmann-Schwinger equation

c) Moller operators and trace class perturbation (preprint).

- Gell-Mann (15/07/1928→24/05/2019) and Goldberger, The Formal Theory of Scattering, 53'.
Abstract : The theory of scattering is developed from first principles with strict attention to the question of the preparation of the state vector of the system appropriate to a description of scattering. The connection between the present formulation and the more conventional interaction representation and S matrix presentations is traced. The wave matrix of Møller is introduced and the existence of bound states is discussed in connection with it. A number of applications to rather involved processes are discussed. Finally, the problem of self-energies in field-theoretic scattering calculations is treated.

- Lax-Phillips 67'.
- Reed-Simon Methods of modern mathematical physics : III Scattering theory, 79'.
- Kato Perturbation Theory for Linear Operators, 80'.

Decomposition of Hilbert space X into orthogonal sum of invariant subspaces

$$X = X_0^{\mathrm{ac}} \oplus X_0^{\mathrm{sc}} \oplus X_0^{\mathrm{pp}} \tag{2}$$

where X_0^{ac} (resp. X_0^{sc}, resp. X_0^{pp}) corresponds to the absolutely continuous (resp. singular continuous, resp. pure point) part of the spectrum.

Mutatis mutandis, the same characterization hold also for H.

Definition

Let P_0 be the projection operator onto the X_0^{ac}. The limit $W_\pm$ (if it exists) is called the Møller wave operator

$$W_\pm = \overset{\text{strong}}{\lim_{t \to \pm\infty}} e^{itH} e^{-itH_0} P_0$$

If ran $W_\pm = X^{\mathrm{ac}}$, then $W_\pm$ is said to be complete.

Kato-Birman theory brings the trace class criterion for a compact operator T

$$\|T\|_1 := \sum_{j \geq 0} \sqrt{\lambda_j(T^* T)} < \infty \tag{3}$$

where the $(\lambda_j)_{j \in \mathbb{N}}$ are the eigenvalues of the compact operator $T^* T \geq 0$.

Theorem (Kato-Birman)

Let $T = (H - z)^{-n} - (H_0 - z)^{-n}$ be trace-class for $n \geq 1$ and z with $Im\ z \neq 0$. Then the wave operators $W_\pm(H, H_0)$ exist and are complete.

Vlasov-Poisson is "almost" trace-class

Consider the homogeneous case treated Fourier mode per Fourier mode.

Lemma

Boltzmanian profile homogeneous in space, $E_0 := 0$ and $M := \exp(-v^2/4)$. Then, for all $n \in \mathbb{N}^*$, the operator $T = (H-z)^{-n} - (H_0 - z)^{-n}$ is not trace class.

The complex number $z \notin \mathbb{R}$ is arbitrary, take $z = i\beta$ with $\beta \in \mathbb{R}^*$.
Note $T = (H-z)^{-n} - (H_0 - z)^{-n}$.
Fourier decomposition (mode $k \in \mathbb{Z}$) with natural notations yields

$$\lim_{|k|\to\infty} \left(k^2 |z|^{2n} \frac{\|T^k u_k\|^2}{\|u_k\|^2} \right) = \sqrt{2\pi}.$$

where u_k is the zero eigenvector $H_k u_k = 0$.

One gets a logarithmic divergence

$$\|T\|_1 \gtrsim \frac{(2\pi)^{\frac{1}{4}}}{|z|^n} \sum_{k \neq 0} \frac{1}{|k|} = +\infty$$

T is not globally trace-class, but the $(T^k)_{k\in\mathbb{Z}}$ are trace-class individually because the perturbation is finite rank. **T is "almost" trace-class.**

Back to electron hole

Claims

i) The trace-class property holds and the wave operators exist.
ii) With structural assumptions and rescaling $\varphi_0 \leftarrow \varepsilon\varphi_0$ (similar to Born expansion), linear Landau damping holds for the model problem.

Structural assumptions for electric potential φ_0.

- The potential φ_0 has regularity $W^{4,\infty}(\mathbb{T})$.
- It is increasing from 0 to $x^* \in (0,1)$, and decreasing from $x^* \in (0,1)$ to 1 : tit is **one bump**

$$\varphi_0(0) = \varphi_0(1) = 0 \text{ and } \varphi_0^+ = \varphi_0(x^*) > 0.$$

- One has $\varphi_0''(0) > 0$ and $\varphi_0''(x^*) < 0$.
- The time spent by a particle which moves along closed characteristics is a monotone function of the label of the characteristics, **even in the hole.**

  A sufficient condition is

$$\frac{d^2}{dx^2}\sqrt{\varphi_0(x^*) - \varphi_0(x)} < 0 \quad x \neq x^*.$$

- D., Scattering structure and Landau damping for linearized Vlasov eq. with inhomogeneous Boltzmannian states, Ann. IHP.

- D., Trace class properties of the linear Vlasov-Poisson equation, preprint.

Morrison integral operator= generalized Moller wave operator

Here $E_0 = \varphi_0 = 0$. Consider the integro-differential operator L (rewriting of the $\mathcal{G}$ transform Morrison 00')

$$Lu = \left(-\partial_{xx} + q(v)\right) u(x,v) - vM(v) P.V. \int_{\mathbb{R}} \frac{1}{w-v} u(x,w) M(w) dw$$

where the function q is defined by

$$q(v) = P.V. \int_{\mathbb{R}} \frac{w}{w-v} M(w)^2 dw.$$

Take u that satisfies $\partial_t u + v\partial_x u + vME = 0$ with the Gauss law.
Then $h = Lu$ is a solution of free transport $\partial_t h + v\partial_x h = 0$.

- Morrison, Hamiltonian Description of Vlasov Dynamics : Action-Angle Variables for the Continuous Spectrum, 2000
- Després, Symmetrization of Vlasov-Poisson equations, 2014.

In other words $u(t) = L^{-1} e^{-v\partial_x t} L$, that is

$$e^{iHt} = L^{-1} e^{iH_0 t} L, \quad iH_0 = -v\partial_x.$$

Intertwinning property $LH = H_0 L$ (generalized wave operators, Yafaev 01).

- Limitation so far for scattering structures : rigorous results only 1D.