Numerical results for the 1d-2v Vlasov-Poisson system in a uniform magnetic field

Alexandre Rege[1]
Frédérique Charles[1]
Bruno Després[1]

alexandre.rege@sorbonne-universite.fr

(1) Laboratoire Jacques-Louis Lions, Sorbonne Université, CNRS UMR 7598

Background

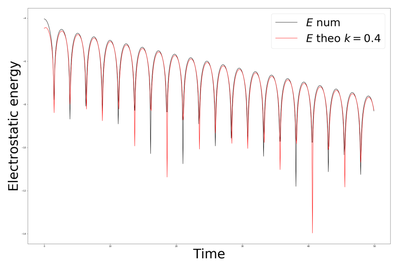

Fig 0 - Landau Damping for 1d-1v Vlasov-Poisson system

➢ According to linear theory, wave solutions of the unmagnetized Vlasov equation exhibit collisionless damping, while in a magnetized plasma, waves perpendicular to the magnetic field are exactly undamped, no matter how weak the magnetic field is, this is the **Landau-Bernstein paradox**.

➢ The theory for magnetized plasma [2] does not converge smoothly to the theory for unmagnetized plasmas when the magnetic field goes to zero [7].

➢ Our general goal is to develop mathematical and numerical analysis of this phenomenon. The poster is essentially restricted to the presentation of numerical results obtained with a semi-lagrangian method.

1. The model

Vlasov-Poisson system in dimension 1d-2v

$$\begin{cases} \partial_t f + v_1 \partial_x f - E \partial_{v_1} f + \omega_c(-v_2 \partial_{v_1} f + v_1 \partial_{v_2} f) = 0 \\ \partial_x E = 1 - \int_{(v_1,v_2)\in\mathbb{R}^2} f(x, v_1, v_2) dv_1 dv_2 \end{cases} \tag{1}$$

Here $\omega_c > 0$ is the cyclotron frequency for electrons and the unknowns are the density of electrons $f(t, x, v_1, v_2)$ and the electric field $E(t, x)$. The domain is $\Omega = \mathbb{T} \times \mathbb{R}^2, \mathbb{T} = [0\,; L]_{per}$ is the 1D-torus.

Assumptions

➢ The total mass is normalized $\int_{\mathbb{T}\times\mathbb{R}^2} f_0 dx dv_1 dv_2 = 1$ (and the Vlasov equation guarantees that the total mass is conserved).

➢ The mean value of the electric field vanishes $\int_{\mathbb{T}} E(t, x) dx = 0$.

2. Semi-Lagrangian scheme with splitting

Principle of the classical (backward) Semi-lagrangian method [3]

The aim is to find an approximation f_n of the solution of $\partial_t f + E(x, t)\partial_x f = 0$ at all discrete time t_n.

➢ For every point x_i of the grid in x, we compute the value of the characteristic which is equal to x_i at time t_{n+1}.

➢ We compute f_{n+1} by interpolation using these values and f_n.

Splitting

This system is particularly adapted to a numerical scheme that uses splitting operations: we can split the Vlasov equation in three equations with constant advection terms. These equations write as:

$$\partial_t f + v_1 \partial_x f = 0 \tag{2}$$
$$\partial_t f - (E + \omega_c v_2)\partial_{v_1} f = 0 \tag{3}$$
$$\partial_t f + \omega_c v_1 \partial_{v_2} f = 0 \tag{4}$$

Algorithm to solve system (1)

1. **Initialisation**

➢ We are given the initial distribution function $f_0(x, v)$, from which we can compute the initial electron density $\rho(0, x) = \int_{\mathbb{R}} f_0(x, v) dv$.

➢ We then compute the initial electric field $E(0, x)$ by solving the Poisson equation $\partial_x E = 1 - \rho(0, x)$.

2. **Going from** t_n **to** t_{n+1}

We assume that we know the matrix $(f^n(x_i, v_j))$, whose coefficients are simply the values of the approximation of $f(t_n, \cdot, \cdot)$ at the grid points (x_i, v_j), and E^n, the approximation of the electric field at time t_n, at the grid points x_i of the physical space.

➢ We compute f^* by solving

$$\partial_t f + v_1 \partial_x f = 0$$

during one time step Δt with initial condition f^n.

➢ We update the electric field by computing $\rho^{n+1}(x) = \int_{\mathbb{R}} f^*(x, v) dv$ and solving the Poisson equation $\partial_x E^{n+1} = 1 - \rho^{n+1}(x)$.

➢ We compute f^{**} by solving

$$\partial_t f - (E^{n+1} + \omega_c v_2)\partial_{v_1} f = 0$$

during one time step Δt with initial condition f^*.

➢ We compute f^{n+1} by solving

$$\partial_t f + \omega_c v_1 \partial_{v_2} f = 0$$

during one time step Δt with initial condition f^{**}.

3. Numerical results: illustration of the Landau Bernstein paradox

Interpolation

➢ We use a classical cubic spline interpolation with periodic boundary conditions (in all three variables).

➢ Cubic spline interpolation is conservative, which ensures that the algorithm conserves mass.

Numerical parameters

➢ In the following simulations, we use the algorithm from section 2 with $L = \frac{2\pi}{k}$ and the initial condition f_0 given by:

$$f_0(x, v_1, v_2) = \frac{1}{2\pi}(1 + \epsilon \cos(kx)) \exp(-\frac{v_1^2 + v_2^2}{2}) \tag{5}$$

with $\epsilon = 0.001$.

➢ This is the classical initial condition used for testing Landau-Damping.

Evolution of the electrical energy in time

➢ For Fig. 1-3, green lines indicate the multiples of $\frac{L}{\Delta v_1}$.

➢ For Fig. 4-6, green lines indicate the multiples of $\frac{2\pi}{\omega_c}$.

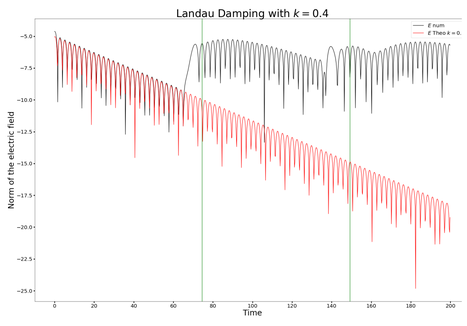

Fig 1 - Numerical recurrence $\omega_c = 0$

Fig 2 - Physical recurrence hidden by numerical recurrence $\omega_c = 0.01$

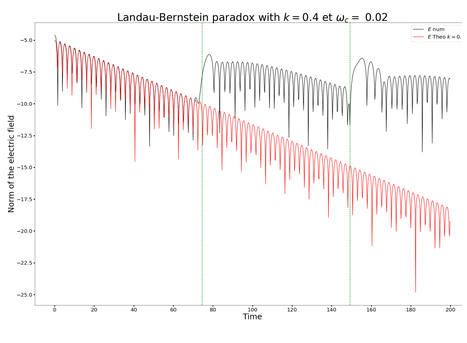

Fig 3 - Physical recurrence hidden by numerical recurrence $\omega_c = 0.02$

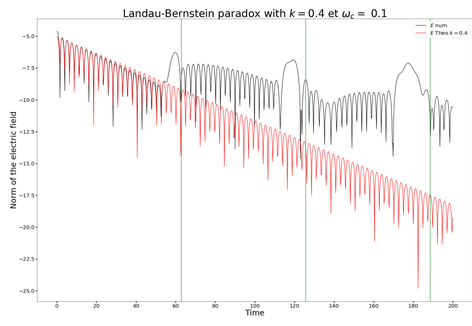

Fig 4 - Physical recurrence $\omega_c = 0.1$

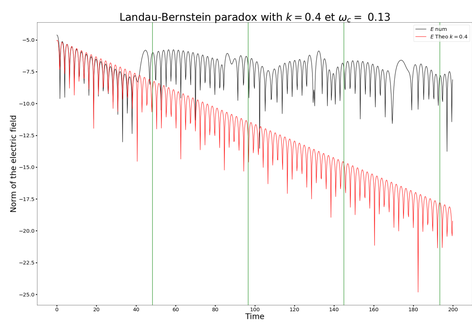

Fig 5 - Physical recurrence $\omega_c = 0.13$

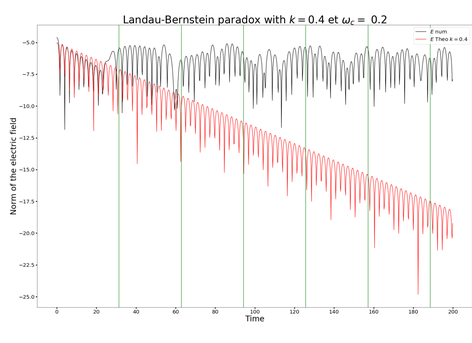

Fig 5 - Physical recurrence $\omega_c = 0.2$

Numerical recurrence vs physical recurrence

➢ The drawback of semi-lagrangian schemes is the appearance of a numerical recurrence [4], [6].

➢ Time of numerical recurrence $T_N \approx \frac{L}{\Delta v_1}$ and time of physical recurrence $T_P \approx \frac{2\pi}{\omega_c}$

➢ We see that for small values of ω_c (Fig 1 and 2), the physical recurrence is hidden by the numerical recurrence.

➢ The physical recurrence is visible when ω_c is "big enough" (Fig 1 and 2) or more precisely verifies $\frac{2\pi}{\omega_c} < \frac{L}{\Delta v_1}$

4. Prospects

➢ We linearize system (1) around the stationary solution $(f_0, E_0) = (\frac{1}{2\pi} e^{-\frac{v_1^2+v_2^2}{2}}, 0)$ to get:

$$\begin{cases} \partial_t u + v_1 \partial_x u - F v_1 e^{-\frac{v_1^2+v_2^2}{4}} u + \omega_c(-v_2 \partial_{v_1} u + v_1 \partial_{v_2} u) = 0 \\ \partial_x F = -\int_{(v_1,v_2)\in\mathbb{R}^2} u(x, v_1, v_2) e^{-\frac{v_1^2+v_2^2}{4}} dv_1 dv_2 \end{cases} \tag{6}$$

➢ Write the linearized system in the form $\partial_t u = iHu$ with H a symmetric operator (framework of scattering theory) and conduct a spectral study of H.

➢ With this framework, find analytic solutions of the problem to further study the L-B paradox and the situation $\omega_c \to 0$.

➢ Compare with a 3d-3v analysis developed in [1].

➢ Compare the theoretical and numerical results: eigenvectors, eigenvalues, decay rate.

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