A relativistic particle pusher for ultra-strong electromagnetic fields

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The context

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❖ Our numerical scheme
❖ Overview of the algorithm
❖ Solution to the relativistic equation of motion
❖ Cross electric and magnetic fields
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High-energy astrophysics deals with

- compact stars: black holes, neutron stars.
- strong electro-gravito-magnetic fields.
- relativistic particles.
- high energy radiation: X-rays, gamma-rays.

Need to accurately solve for realistic physical conditions

- fluid motion: RMHD
- kinetic plasma effects
- radiation field.

Resort to numerical simulations but ....

The problem

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- Basic neutron star parameters
  - ✦ magnetic field strength: $B = 10^5 - 10^8$ T.
  - ✦ rotation period: $P = 1\,ms - 10\,s$.
  - ✦ electric field strength: $E = \Omega\,B\,R = 10^{13}$ V/m.
- Two important frequencies
  - ✦ neutron star rotation frequency $\Omega = 2\,\pi/P = 1 - 10^3$ rad/s.
  - ✦ electromagnetic wave frequency $\omega = \frac{e\,B}{m_{\rm e}} = 10^{16} - 10^{19}$ rad/s.
- The problem with fully kinetic simulations, the strength parameter

$$a = \frac{\omega}{\Omega} = \frac{e\,B}{m_{\rm e}\,\Omega} = 2{,}8 \cdot 10^{18} \left(\frac{P}{1\ \mathrm{s}}\right) \left(\frac{B}{10^8\ \mathrm{T}}\right) \gg 1.$$

$\Longrightarrow$ impossible to follow individual particle on a timescale P.

$\Longrightarrow$ use an analytical pusher not explicitly resolving for the gyration frequency.

Our numerical scheme

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Solve analytically the equation in uniform electromagnetic field in the frame where **E** and **B** are parallel (along z) with the electromagnetic field tensor F^{ik}

$$\frac{du^i}{d\tau} = \frac{q}{m} F^{ik} u_k$$

$$F^{ik} = \begin{pmatrix} 0 & 0 & 0 & -E_z/c \\ 0 & 0 & -B_z & 0 \\ 0 & B_z & 0 & 0 \\ E_z/c & 0 & 0 & 0 \end{pmatrix}.$$

Trajectories depend solely on two relativistic electromagnetic invariants

- $\mathcal{I}_1 = E^2 - c^2 B^2$.
- $\mathcal{I}_2 = \mathbf{E} \cdot \mathbf{B}$.

Overview of the algorithm

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1. If $\mathcal{I}_2 = \mathbf{E} \cdot \mathbf{B} = 0$ meaning **E** and **B** perpendicular
   - if $\mathcal{I}_1 \neq 0$, frame where either **E** or **B** vanishes exists, depending on the sign of $\mathcal{I}_1$

   $\Longrightarrow$ switch to this new frame K' and solve analytically the equation of motion.

   - If $\mathcal{I}_1 = 0$, solve the motion separately as no physical frame K' exist with speed strictly less than c where **E** and **B** are parallel

   $\Longrightarrow$ called a null or light like field.

2. If $\mathcal{I}_2 \neq 0$ a frame K' where **E** and **B** are parallel always exists

   $\Longrightarrow$ switch to the new frame K' by a Lorentz boost

   $\Longrightarrow$ apply an Euler rotation to bring the new z axis along common **E**/**B** direction.

   $\Longrightarrow$ solve the particle motion in K'

   $\Longrightarrow$ Lorentz boost back to K

Solution to the relativistic equation of motion

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For **E** and **B** parallel

$$c\,(t - t_0) = \frac{\gamma_0\,c}{\omega_E}\,\left[\mathrm{sh}(\omega_E\,\tau) + \beta_0^z\,(\mathrm{ch}(\omega_E\,\tau) - 1)\right]$$

$$x - x_0 = \frac{\gamma_0\,c}{\omega_B}\,\left[\beta_0^x\,\sin(\omega_B\,\tau) - \beta_0^y\,(\cos(\omega_B\,\tau) - 1)\right]$$

$$y - y_0 = \frac{\gamma_0\,c}{\omega_B}\,\left[\beta_0^x\,(\cos(\omega_B\,\tau) - 1) + \beta_0^y\,\sin(\omega_B\,\tau)\right]$$

$$z - z_0 = \frac{\gamma_0\,c}{\omega_E}\,\left[(\mathrm{ch}(\omega_E\,\tau) - 1) + \beta_0^z\,\mathrm{sh}(\omega_E\,\tau)\right]$$

For null or light like fields

$$c\,(t - t_0) = \gamma_0\,c[\tau + (1 - \beta_0^x)\,\frac{\omega_B^2\,\tau^3}{6} + \beta_0^y\,\frac{\omega_B\,\tau^2}{2}]$$

$$x - x_0 = \gamma_0\,c\,[\beta_0^x\,\tau + (1 - \beta_0^x)\,\frac{\omega_B^2\,\tau^3}{6} + \beta_0^y\,\frac{\omega_B\,\tau^2}{2}]$$

$$y - y_0 = \gamma_0\,c\,[\beta_0^y\,\tau + (1 - \beta_0^x)\,\frac{\omega_B\,\tau^2}{2}]$$

$$z - z_0 = \gamma_0\,v_0^z\,\tau$$

Cross electric and magnetic fields

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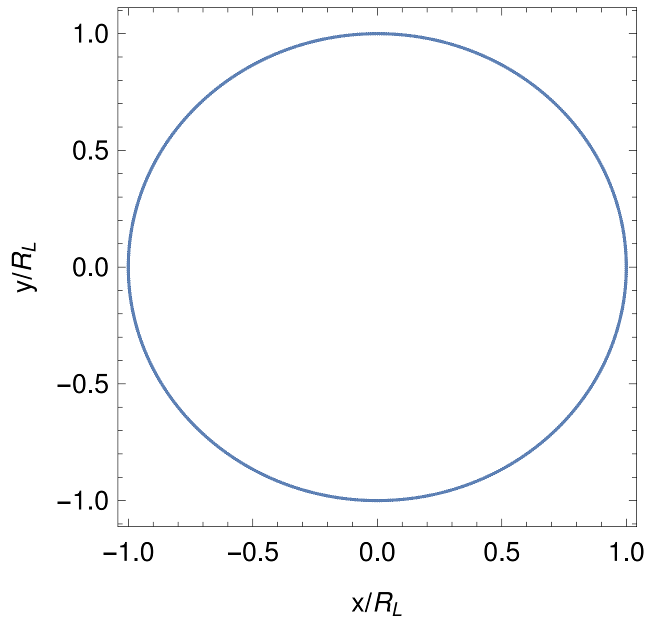


Figure 1: Gyromotion of an electron in the electric drift frame with $\Gamma_E = 10^3$ and $\gamma = 10^{10}$. The Larmor radius is $R_L = 10^{10}$.

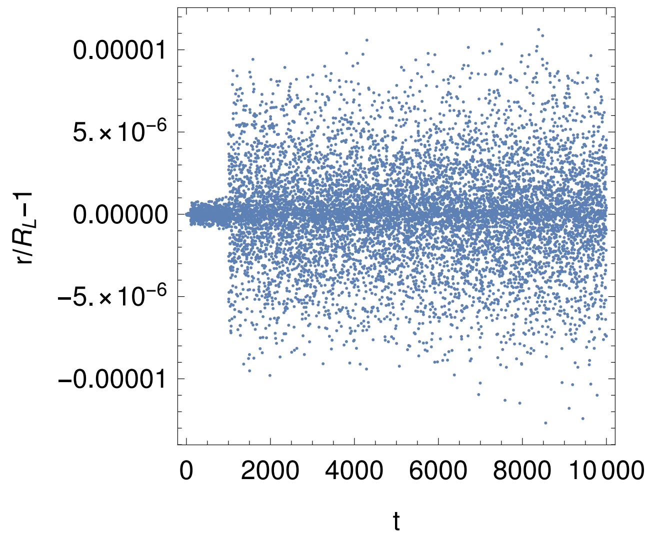
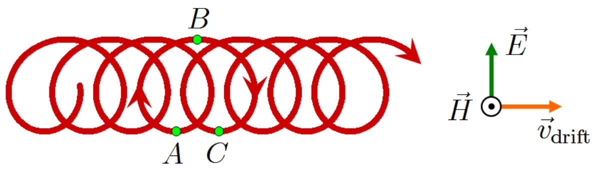


Figure 2: Relative error in the Larmor radius.



Central electric force

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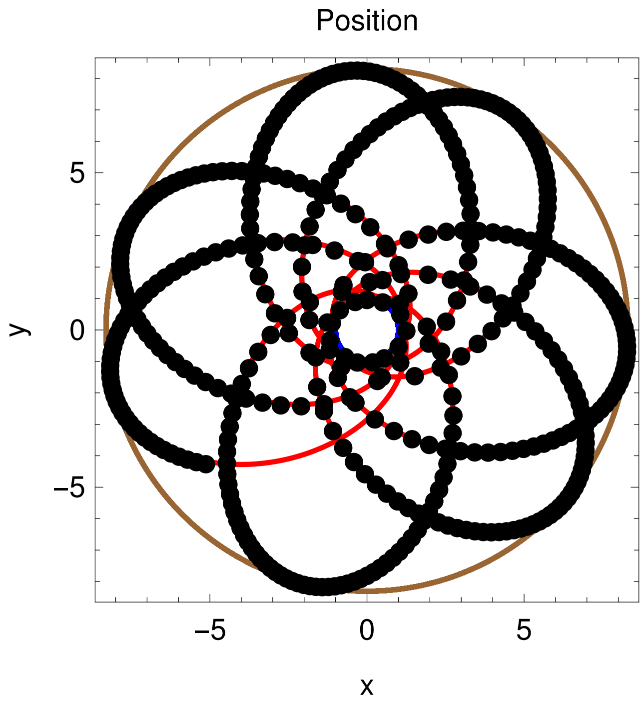

Figure 3: Motion of an electron in the electric field of a fixed proton, black points. Exact analytical solution shown in red.

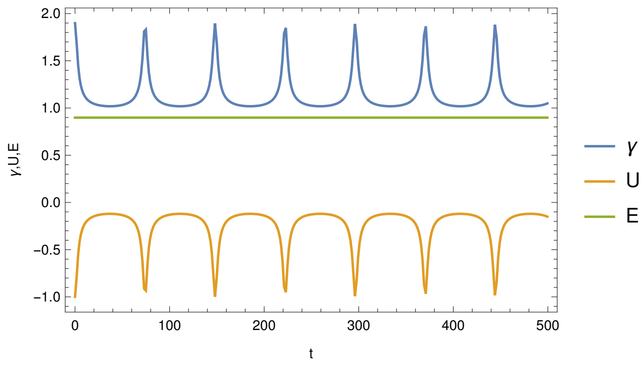

Figure 4: Total energy E, relativistic kinetic energy $\gamma\, m\, c^2$ and electrostatic potential energy U.

Linearly polarized plane wave

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Analytical solution given with respect to the wave phase $\xi = \omega\, t - k\, x$ for a particle at rest at initial time $t = 0$.

$$u^x = \frac{a^2}{2}\, c\, (\cos\xi - 1)^2$$
$$u^y = -a\, c\, (\cos\xi - 1)$$
$$u^0 = c + u^x .$$

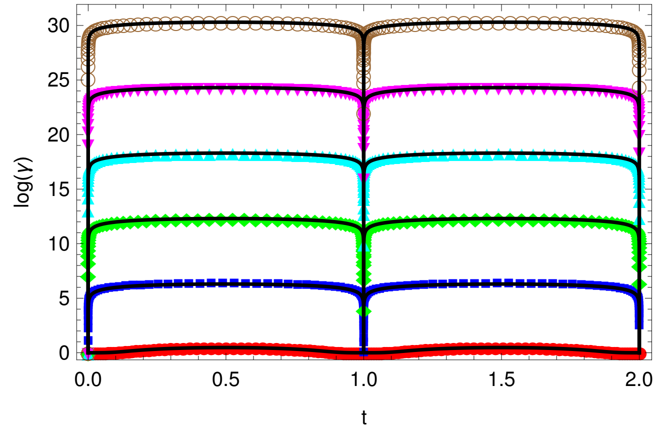

Figure 5: Lorentz factor of an electron for $a = 10^i$ with $i \in \{0, 3, 6, 9, 12, 15\}$.

- Maximum Lorentz factor $\gamma_{\max} = 1 + 2\, a^2$.
- Perfectly period motion with particle returning to rest after a period $P = 2\,\pi\,(1 + 3\, a^2/4)$.
- Ultrarelativistic particles if $a \gg 1$.
- Nonrelativistic particles if $a \ll 1$.

Current state of the art already faces severe flaws at $a \gtrsim 100$ which is unacceptable for neutron stars.

Circularly polarized plane wave

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Another analytical solution

$$u^x = a^2\, c\,(1 - \cos\xi) = a\, u^y$$
$$u^y = a\, c\,(1 - \cos\xi)$$
$$u^z = -a\, c\, \sin\xi$$
$$u^0 = c + u^x .$$

- Maximum Lorentz factor $\gamma_{\max} = 1 + 2\,a^2$.
- Perfectly period motion with particle returning to rest after a period $P = 2\,\pi\,(1 + a^2)$.

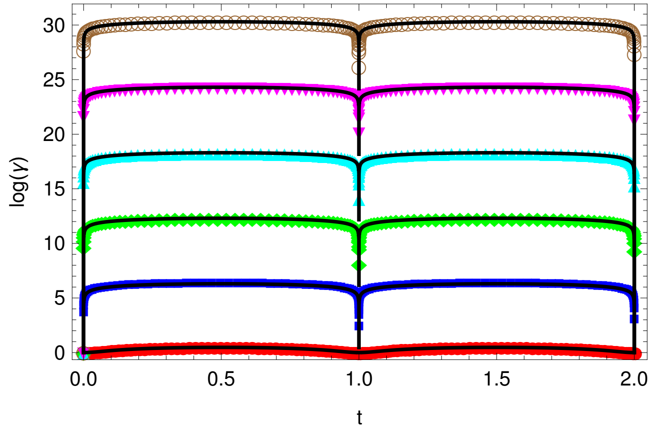

Figure 6: Lorentz factor of an electron for $a = 10^i$ with $i \in \{0, 3, 6, 9, 12, 15\}$.

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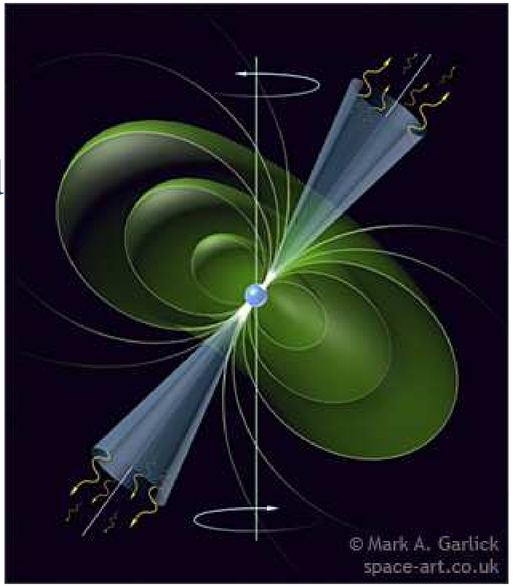
- wave polarisation depending on angle of observation
- particles can escape, hit the star or be trapped
- particles injected at rest will form a power law distribution in energy.
- use exact analytical solution for a rotating magnteic dipole known as Deutsch solution
- magnetic axis inclined with respect to rotation axis by an angle χ



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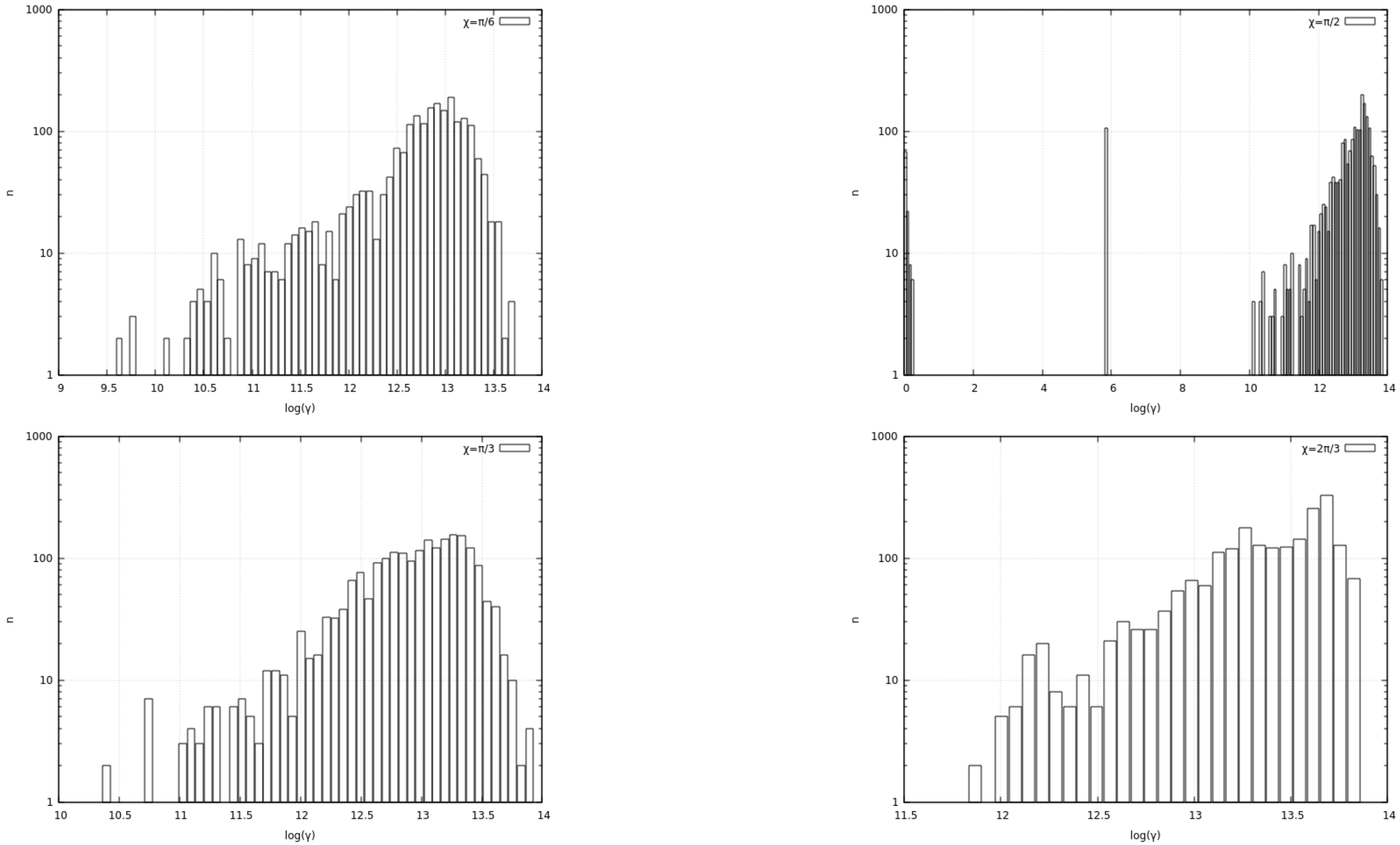
Particle distribution functions for several inclinations $\chi = 30°, 60°, 90°, 120°$.



- Maximum Lorentz factor up to $\gamma \approx 10^{13}$
- too high because radiation reaction sets in and slows down the particles probably to $\gamma \approx 10^9$ (to be confirmed by numerical simulations)

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- new scheme for particle trajectory integration in any electromagnetic field configuration.
- trajectory is given by an explicit close analytical form.
- for spatially and time dependent fields, numerical errors arise from constant field approximation.
- less stringent condition on the time step.
- approaches well suited for neutron star environment.

- look for analytical solutions in fields that are linear in space and time.
- not clear if such analytical solutions are tractable.
- add radiation reaction at such high Lorentz factors.