Study of trapped particles turbulence in magnetic fusion plasmas

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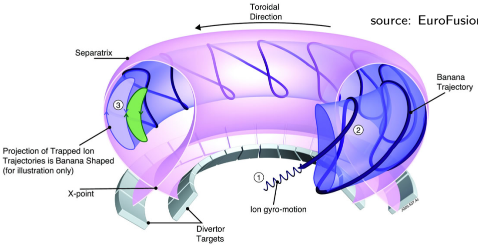
Kinetic model of trapped particles

Trapped particle time scales:
- ⋆ very fast gyromotion Ω_{cs}
- ⋆ fast bounce $\omega_{bs} << \Omega_{cs}$
- ⋆ slow turbulence $\omega << \omega_{bs}$

"Bounce-average gyrokinetics":[1]

$\alpha \approx$ toroidal angle
$\psi \approx$ radius

source: EuroFusion



Low field **High field** **Low field**

Bounce-averaged Vlasov equation:

$$\partial_t F_s + \frac{\Omega_d E}{Z_s}\partial_\alpha F_s - \partial_\psi F_s\,\partial_\alpha \chi_s + \partial_\psi \chi_s\,\partial_\alpha F_s = 0$$

allow to describe Trapped Ion Modes / Trapped Electron Modes

Separate profiles (F_{0s} known) Fourier transf. fluct. ($\delta f_{s\mathbf{k}}$, $\chi_{s\mathbf{k}} = \mathcal{J}_{0s}\phi_{\mathbf{k}}$)

$$\partial_t f_{sk} = \underbrace{ik_\alpha \partial_\psi F_{s0}\chi_{sk}}_{\text{injection}} - ik_\alpha \frac{\Omega_d E}{Z_s} f_{sk} - \underbrace{\sum_{\mathbf{k}+\mathbf{p}+\mathbf{q}=0} [\mathbf{b}\times\mathbf{p}\,.\,\mathbf{q}]\,\chi_{sp}\, f_{sq}}_{\text{transfers}}\,,$$

$$C_k \phi_{\mathbf{k}} = \sum_s Z_s \int_0^{+\infty} \mathcal{J}_{0s} f_{sk}\sqrt{E}\,dE\,. \qquad \textbf{quasi-neutrality}$$

[1] talk by E. Gravier, poster by J. Médina

Conservations: free energy balance[2]

$$\partial_t f_{s\mathbf{k}} = ik_\alpha \partial_\psi F_{s0} \chi_{s\mathbf{k}} - ik_\alpha \frac{\Omega_d E}{Z_s} f_{s\mathbf{k}} - \sum_{\mathbf{k}+\mathbf{p}+\mathbf{q}=0} [\mathbf{b} \times \mathbf{p} \,.\, \mathbf{q}] \, \chi_{s\mathbf{p}} \, f_{s\mathbf{q}} \,,$$

$$C_k \phi_{\mathbf{k}} = \sum_s Z_s \int_0^{+\infty} \mathcal{J}_{0s} f_{s\mathbf{k}} \sqrt{E} dE \,.$$

Entropy balance: $\mathcal{E}_{f_s} = T_{s0} \sum_{\mathbf{k}} \int_0^{+\infty} \frac{|f_{sk}|^2}{2F_{s0}} \sqrt{E} dE$

$$\partial_t \mathcal{E}_{f_s} = \left(\frac{3}{2} \kappa_{T_s} - \kappa_{ns} \right) T_{s0} \Gamma_s - \kappa_{T_s} \mathcal{Q}_s - \mathcal{D}_s$$

Electrostatic energy balance: $\mathcal{E}_\phi = \sum_{\mathbf{k}} C_k \frac{|\phi_{\mathbf{k}}|^2}{2}$

$$\partial_t \mathcal{E}_\phi = -\Omega_d \sum_s \mathcal{Q}_s - \mathcal{D}_\phi$$

Radial fluxes: radial transport due to background gradients

$$\Gamma_s = \sum_{\mathbf{k}} ik_\alpha \phi_{\mathbf{k}} \int_0^{+\infty} \mathcal{J}_{0s}^k f_{s\mathbf{k}}^\star \sqrt{E} dE \,,$$

$$\mathcal{Q}_s = \sum_{\mathbf{k}} ik_\alpha \phi_{\mathbf{k}} \int_0^{+\infty} \mathcal{J}_{0s}^k f_{s\mathbf{k}}^\star E^{3/2} dE \,.$$

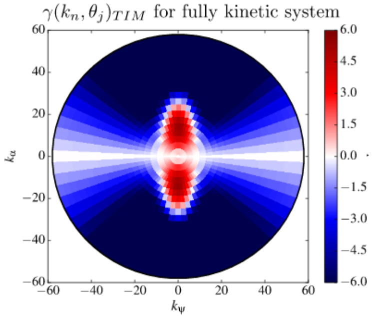
[2] Schekochihin, PPCF 2008, Bañòn Navarro, PRL 2011

TIM and TEM: linear modes

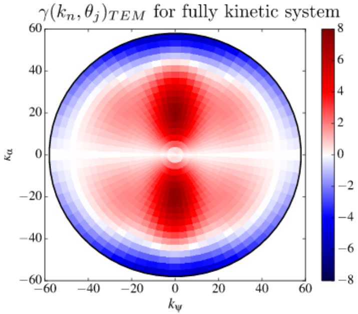
Linear dispersion relation:

$$C_k + \sum_s Z_s \int_0^{+\infty} \frac{\mathcal{J}_{0s}^2 k_\alpha \partial_\psi F_{s0}}{\omega - k_\alpha \Omega_d E / Z_s} \sqrt{E} dE = 0$$

TIM modes:



TEM modes:



- ⋆ $\gamma_{TIM} < \gamma_{TEM}$
- ⋆ TIM peak at largest scales than TEM

anisotropy is important
at least for linear injection

A hierarchy of reduced models

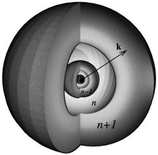
Fourier transform the $\mathbf{E} \times \mathbf{B}$ nonlinearity:

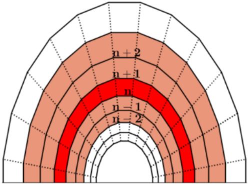
Note: $\mathbf{k} + \mathbf{p} + \mathbf{q} = 0$
form a **triangle**

$$\mathbf{b} \times \nabla\phi \,.\, \nabla f \longrightarrow \sum_{\mathbf{k}+\mathbf{p}+\mathbf{q}=0} [\mathbf{b} \times \mathbf{p} \,.\, \mathbf{q}]\, \phi_{\mathbf{p}} f_{\mathbf{q}}$$

Shell models:

- ⋆ logarithmic k-grid: $k_n = k_0\, g^n$
- ⋆ isotropy of the unknowns: $\phi_{\mathbf{p}} = \phi_p$, $f_{\mathbf{q}} = f_q$
- ⋆ only local interactions: $(p, q) = n \pm 1\,, n \pm 2$





LDM approach:

- ⋆ logarithmic k-grid: $k_n = k_0\, g^n$
- ⋆ keep the angular dependence:
  $\phi_{\mathbf{p}} = \phi\left(p_n, \theta_j\right) = \phi_n^j$
- ⋆ local interactions with triangle conditions

Shell models (GOY/Sabra)

GOY[3]/Sabra[4] models:

- ⋆ logarithmic k-grid: $k_n = k_0\, g^n$
- ⋆ isotropy of the unknowns: $\phi_{\mathbf{p}} = \phi_p$, $f_{\mathbf{q}} = f_q$
- ⋆ only local interactions: $(p, q) = n \pm 1\,, n \pm 2$

Sabra version for TEM/TIM turbulence:[5]

$$\begin{aligned}\partial_t f_{sn} \quad = \quad & ik_n \chi_{sn} \partial_\psi F_{s0} - ik_n \frac{\Omega_d E}{Z_s} f_{sn}^{s,j} + \alpha \frac{k_n^2}{g} \left[g^{-2} \left(\chi_{s,n-2} f_{s,n-1} - \chi_{s,n-1} f_{s,n-2} \right) \right. \\ & \left. - \left(\chi^\star_{s,n-1} f_{s,n+1} - \chi_{s,n+1} f^\star_{s,n-1} \right) + g^2 \left(\chi^\star_{s,n+1} f_{s,n+2} - \chi_{s,n+2} f^\star_{s,n+1} \right) \right]\end{aligned}$$

- ⋆ source: equilibrium gradients $\qquad \partial_\psi F_{s0} \propto \frac{\nabla n_s}{n_s} + \left(E - \frac{3}{2} \right) \frac{\nabla T_s}{T_s}$
- ⋆ advection with precession frequency Ω_d
- ⋆ non-linear couplings: n with $\{n \pm 1; n \pm 2\}$

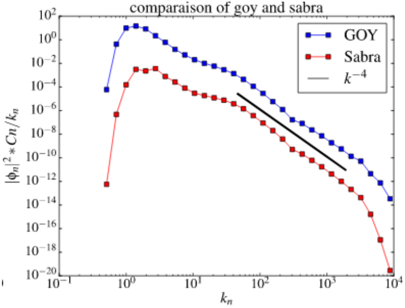
Note the complex conjugates *

GOY version: CC everywhere

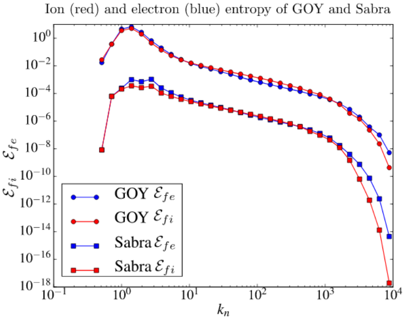
[3] Gledzer, Dokl. Akad. Nauk SSSR (1973), and Yamada, Ohkitani, J. Phys. Soc. Jpn (1987)
[4] L'vov, *et al*, PRE (1998).
[5] Shaokang Xu, PoP (2018)

GOY/Sabra: spectrae[6]

Electrostatic energy $\mathcal{E}_\phi$:



Entropy $\mathcal{E}_{f_s}$:



- ⋆ similar slopes:

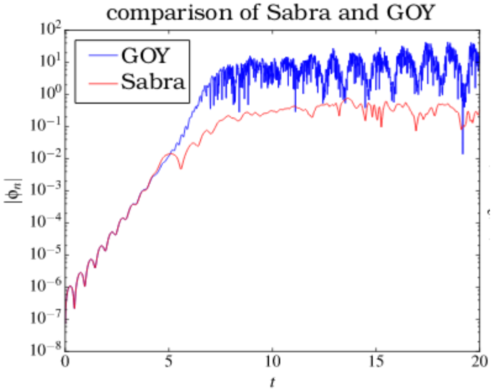
$$\mathcal{E}_\phi \sim k^{-4}$$
$$\mathcal{E}_{f_s} \sim k^{-1}$$

- ⋆ Saturation levels differ from ≈ 3 decades ?

[6] Shaokang Xu, PoP 2018a, & R. Lustrat, M2 internship report

GOY/Sabra: time traces[7]

Electrostatic energy $\mathcal{E}_\phi$:



- ⋆ very different time behaviors !
- ⋆ GOY displays oscillations
- ⋆ Sabra is far more chaotic
- ⋆ only difference: phases

  GOY:
  $\partial_t f|_{NL} \propto \chi^\star_{s,n-1} f^\star_{s,n+1}$
  Sabra:
  $\partial_t f|_{NL} \propto \chi^\star_{s,n-1} f_{s,n+1}$

$\Rightarrow$ phase dependence matters!
Link with isotropy assumption

[7] Shaokang Xu, PoP 2018a

LDM equation for TEM/TIM[8]

- ⋆ logarithmic k-grid: $k_n = k_0\, g^n$ + **regular θ_k-grid:** $\theta^{\mathbf{j}} = \mathbf{j}\frac{2\pi}{\mathbf{M}_\theta}$
- ⋆ keep the angular dependence: $\phi_{\mathbf{p}} = \phi\left(p_n, \theta_j\right) = \phi_n^j$
- ⋆ local interactions: n coupled to $n \pm 1$, $n \pm 2$

$$
\begin{aligned}
\partial_t f_{\ell,n}^{s,j} \quad = \quad & i k_\alpha \chi_{\ell,n}^{s} \partial_\psi F_{eq}^{s} - i k_\alpha \frac{E_\ell \Omega_d}{Z} f_{\ell,n}^{s,j} \\
& + \frac{k_n^2 g^{-4}}{2} \sqrt{\mu_0} \left[\chi_{\ell,n-2}^{\star \mathbf{j}+\mathbf{r_0}} f_{\ell,n-1}^{\star \mathbf{j}-\mathbf{s_0}} - \chi_{\ell,n-1}^{\star \mathbf{j}-\mathbf{s_0}} f_{\ell,n-2}^{\star \mathbf{j}+\mathbf{r_0}} + \chi_{\ell,n-1}^{\star \mathbf{j}+\mathbf{s_0}} f_{\ell,n-2}^{\star \mathbf{j}-\mathbf{r_0}} - \chi_{\ell,n-2}^{\star \mathbf{j}-\mathbf{r_0}} f_{\ell,n-1}^{\star \mathbf{j}+\mathbf{s_0}} \right] \\
& + \frac{k_n^2 g^{-2}}{2} \sqrt{\mu_0} \left[\chi_{n-1}^{\star \mathbf{j}+\ell_0} f_{\ell,n+1}^{\star \mathbf{j}-\mathbf{s_0}} - \chi_{\ell,n+1}^{\star \mathbf{j}-\mathbf{s_0}} f_{\ell,n-1}^{\star \mathbf{j}+\ell_0} + \chi_{\ell,n+1}^{\star \mathbf{j}+\mathbf{s_0}} f_{\ell,n-1}^{\star \mathbf{j}-\ell_0} - \chi_{\ell,n-1}^{\star \mathbf{j}-\ell_0} f_{\ell,n+1}^{\star \mathbf{j}+\mathbf{s_0}} \right] \\
& + \frac{k_n^2}{2} \sqrt{\mu_0} \left[\chi_{\ell,n+1}^{\star \mathbf{j}+\ell_0} f_{\ell,n+2}^{\star \mathbf{j}-\mathbf{r_0}} - \chi_{\ell,n+2}^{\star \mathbf{j}-\mathbf{r_0}} f_{\ell,n+1}^{\star \mathbf{j}+\ell_0} + \chi_{\ell,n+2}^{\star \mathbf{j}+\mathbf{r_0}} f_{\ell,n+1}^{\star \mathbf{j}-\ell_0} - \chi_{\ell,n+1}^{\star \mathbf{j}-\ell_0} f_{\ell,n+2}^{\star \mathbf{j}+\mathbf{r_0}} \right]
\end{aligned}
$$

- ⋆ source: equilibrium gradients $\partial_\psi F_{s0} \propto \frac{\nabla n_s}{n_s} + \left(E - \frac{3}{2}\right) \frac{\nabla T_s}{T_s}$
- ⋆ advection with precession frequency Ω_d
- ⋆ non linear couplings n with $\{n \pm 1; n \pm 2\}$, **angular shifts $\pm \mathbf{r_0}$, $\pm \mathbf{s_0}$, $\pm \ell_0$**

[8] Shaokang Xu, PoP 2018b

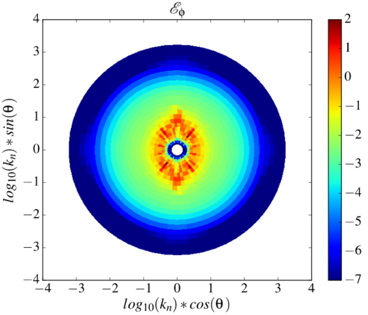
LDM simulations

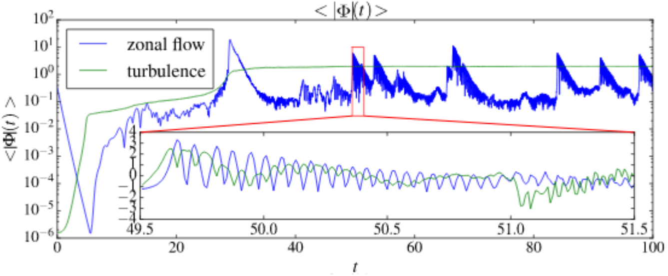
- LDM : $\phi^j_n = \phi\left(k_n, \theta_j\right)$
- streamers $k_\psi = 0$ *vs* Zonal Flows $k_\alpha = 0$ anisotropy
- time trace $\phi(t) = \overline{\phi}_{\mathrm{ZF}}(t) + \widetilde{\phi}(t)$

  intermittent, bursts ?

  predator (ZF) *vs* prey (turb.)

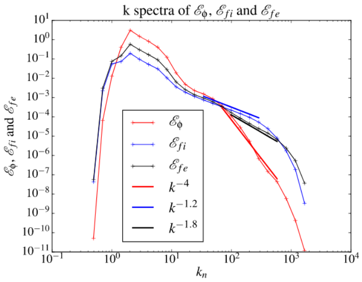
  ZF with local couplings ?

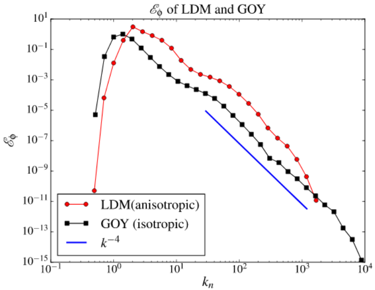




LDM simulations

k-spectrae:





- $\mathcal{E}_\phi \sim k^{-4}$
- $\mathcal{E}_{f_s} \sim k^{-1} - k^{-2}$
- do not depend much on model chosen
- do not depend much on phase dynamics

Discussion

- ⋆ hierarchy of reduced models developed

shell $\longrightarrow$ (spiral $\longrightarrow$) LDM

- ⋆ shell models = GOY or Sabra

similar spectral slopes

saturation level, time trace differ

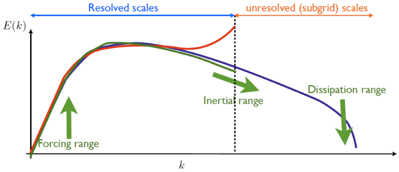
- ⋆ LDM model more accurate

same spectral slopes

more accurate description of Zonal Flows

incorporate non-local couplings ?

- ⋆ Spiral chains: promising intermediate[9]
- ⋆ candidates for improved "sub-grid" models ?



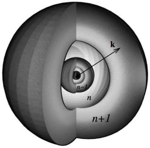
Thank you for your attention

[9] Gürcan et al, subm. PRE

A hierarchy of reduced models

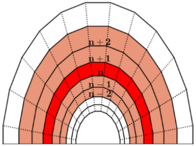
Fourier transform the $\mathbf{E}\times\mathbf{B}$ nonlinearity:

$$\mathbf{b}\times\nabla\phi\,.\,\nabla f \longrightarrow \sum_{\mathbf{k}+\mathbf{p}+\mathbf{q}=0}[\mathbf{b}\times\mathbf{p}\,.\,\mathbf{q}]\,\phi_{\mathbf{p}}f_{\mathbf{q}}$$



Shell models:

- ⋆ logarithmic k-grid: $k_n = k_0\, g^n$
- ⋆ isotropy of the unknowns: $\phi_{\mathbf{p}} = \phi_p,\ f_{\mathbf{q}} = f_q$
- ⋆ only local interactions: $(p,q) = n\pm 1\,,\, n\pm 2$

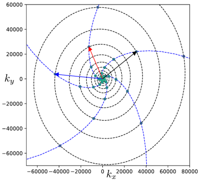


LDM approach:

- ⋆ logarithmic k-grid: $k_n = k_0\, g^n$
- ⋆ keep the angular dependence: $\phi_{\mathbf{p}} = \phi\left(p_n,\theta_j\right) = \phi_n^j$
- ⋆ local interactions with triangle conditions

Spiral chains:

- ⋆ spiral k-grid: $\mathbf{k}_n = k_0 g^n e^{in\varphi}$
- ⋆ form exact triangles
- ⋆ relax the locality contraint $n\pm 1\,,\ n\pm 3$



Differential approximation: Motivation

- $\mathbf{E} \times \mathbf{B}$ nonlinearity:

$$\left.\frac{\partial f}{\partial t}\right|_{\mathrm{NL}} = \sum_{\mathbf{k}+\mathbf{p}+\mathbf{q}=\mathbf{0}} [\mathbf{z} \times \mathbf{p}] \cdot \mathbf{q} \;\; \chi_{\mathbf{p}} f_{\mathbf{q}}$$

universal in plasmas

same form in 2D fluid turbulence (stream function ψ)

- Conserved quantities:

$$\mathcal{E}_f \propto f^2 \,, \quad \mathcal{E}_\phi \propto C_k \phi^2 \,.$$

- Find a model allowing to write energies spectral budgets:

$$\begin{aligned} \partial_t \mathcal{E}_k^{f_s} + \partial_k \Pi_k^{f_s} &= \mathcal{G}_k^{f_s} - \mathcal{D}_k^{f_s} \\ \partial_t \mathcal{E}_k^{\phi} + \partial_k \Pi_k^{\phi} &= \mathcal{G}_k^{\phi} - \mathcal{D}_k^{\phi} \end{aligned}$$

Π_k^f, Π_k^ϕ are spectral energy fluxes

play a key role in turbulence

approximation for $\Pi_k^{f,\phi}$?

Differential approximation: algebra

Assume:

$$k_n = k_0\, g^n = k_0\,(1+\epsilon)^n \ ,$$

with ϵ small, at fourth order:

$$\begin{aligned} k_{n+1} &= k_n + \epsilon k_n\,, \\ f_{n+1} &\approx f_n + \epsilon k_n \partial_k f_n + \frac{\epsilon^2 k_n^2}{2}\partial_k^2 f_n + \frac{\epsilon^3 k_n^3}{6}\partial_k^3 f_n + \frac{\epsilon^4 k_n^4}{24}\partial_k^4 f_n\,, \\ \ldots & \quad \ldots \end{aligned}$$

Plug into GOY/Sabra truncation $n \pm 1$, $n \pm 2$:

$$\begin{aligned} N &\approx \alpha \frac{k}{\chi}\,\partial_k \left[k^2 \chi^{3/2}\,\partial_k \left(k^2 \chi^{3/2} \partial_k \frac{f}{\chi}\right)\right] \\ &\approx -\alpha \frac{k}{f}\,\partial_k \left[k^2 f^{3/2}\,\partial_k \left(k^2 f^{3/2} \partial_k \frac{\chi}{f}\right)\right] \ , \end{aligned}$$

- ⋆ Antisymetric: $N\,[f,\chi] = -N\,[\chi, f]$ Poisson bracket
- ⋆ Construct energies: $N \times (\chi/k)$ or $N \times (f/k)$
- ⋆ no phase involved: χ and f are amplitudes

Passive scalar equations

$$
\begin{aligned}
\partial_t \nabla^2 \phi + \mathbf{z} \times \nabla\phi \,.\, \nabla\nabla^2\phi &= 0,\,, \\
\partial_t n + \mathbf{z} \times \nabla\phi \,.\, \nabla n &= 0\,.
\end{aligned}
$$

Differential approximation:

$$
\begin{aligned}
\partial_t k^2 \phi &= 2\alpha \frac{k}{\phi} \partial_k \left[k^2 \phi^{3/2} \partial_k \left(k^3 \phi^{3/2} \right) \right] \,, \\
\partial_t n &= -\alpha \frac{k}{n} \partial_k \left[k^2 n^{3/2} \partial_k \left(k^2 n^{3/2} \partial_k \frac{\phi}{n} \right) \right] \,.
\end{aligned}
$$

Energy formulation: with injection $\mathcal{I}$ and dissipations $\nu_s^{\phi,n}$, $\nu_L^{\phi,n}$

$$
\begin{aligned}
\partial_t \mathcal{E}_{\phi,k} &= 2\alpha \partial_k \left[k^{5/4} \mathcal{E}_{\phi,k}^{3/4} \partial_k \left(k^{9/4} \mathcal{E}_{\phi,k}^{3/4} \right) \right] \\
&\quad + \mathcal{I}_{\phi,k} - \nu_s^{\phi} k^4 \mathcal{E}_{\phi,k}^{1/2} - \nu_L^{\phi} k^{-6} \mathcal{E}_{\phi,k} \,, \qquad (1) \\
\partial_t \mathcal{E}_{n,k} &= -\alpha \partial_k \left[\frac{\mathcal{E}_{n,k}^{3/4}}{k^{-11/4}} \partial_k \left(\frac{\mathcal{E}_{n,k}^{3/4}}{k^{-11/4}} \partial_k \frac{\mathcal{E}_{\phi,k}^{1/2}}{k \mathcal{E}_{n,k}^{1/2}} \right) \right] \\
&\quad + \mathcal{I}_{n,k} - \nu_s^{n} k^4 \mathcal{E}_{n,k} - \nu_L^{n} k^{-6} \mathcal{E}_{n,k} \,, \qquad (2)
\end{aligned}
$$

Passive scalar: spectrae

- Vorticity stationnarity:

$$\partial_k \left[k^{5/4} \mathcal{E}_{\phi,k}^{3/4} \partial_k \left(k^{9/4} \mathcal{E}_{\phi,k}^{3/4} \right) \right] \approx 0$$

$\mathcal{E}_\phi \sim k^{-3}$ or $\mathcal{E}_\phi \sim k^{-5/3}$

recover Kraichnan-Kolmogorov

- Note:

$$\partial_k \left[k^{5/4} \mathcal{E}_{\phi,k}^{3/4} \partial_k \left(k^{9/4} \mathcal{E}_{\phi,k}^{3/4} \right) \right] = \partial_k \frac{\partial_k \left(k^{9/2} \mathcal{E}_{\phi}^{3/2} \right)}{k},$$

recover the Leith model

- Passive scalar stationnarity:

$$\partial_k \left[k^{11/4} \mathcal{E}_{n,k}^{3/4} \partial_k \left(k^{11/4} \mathcal{E}_{n,k}^{3/4} \partial_k \frac{\mathcal{E}_{\phi,k}^{1/2}}{k \mathcal{E}_{n,k}^{1/2}} \right) \right] \approx 0$$

Gives six different slopes:

$$\mathcal{E}_{n,k} \sim \left\{ k^{-5} ;\ k^{-11/3} ;\ k^{-5/3} ;\ k^{-1} ;\ k^{1/3} ;\ k^{3} \right\}$$

Signs of the fluxes

Remind the defnition of the fluxes:

$$\partial_t \mathcal{E}_\phi = -\partial_k \Pi_k^\phi \propto \partial_k \frac{\partial_k \left(k^{9/2} \mathcal{E}_\phi^{3/2} \right)}{k},$$

$$\partial_t \mathcal{E}_n = -\partial_k \Pi_k^n \propto -\partial_k \left[k^{11/4} \mathcal{E}_n^{3/4} \partial_k \left(k^{11/4} \mathcal{E}_n^{3/4} \partial_k \sqrt{\frac{\mathcal{E}_\phi}{k^2 \mathcal{E}_n}} \right) \right]$$

Constant, nonzero, fluxes correspond to :

- ⋆ $\Pi_k^\phi < 0$ for $\mathcal{E}_\phi \sim k^{-5/3}$

Inverse energy cascade

- ⋆ $\Pi_k^n > 0$ for $\mathcal{E}_\phi \sim k^{-5/3}$ and $\mathcal{E}_n \sim k^{-5/3}$
- ⋆ $\Pi_k^n > 0$ for $\mathcal{E}_\phi \sim k^{-3}$ and $\mathcal{E}_n \sim k^{-1}$

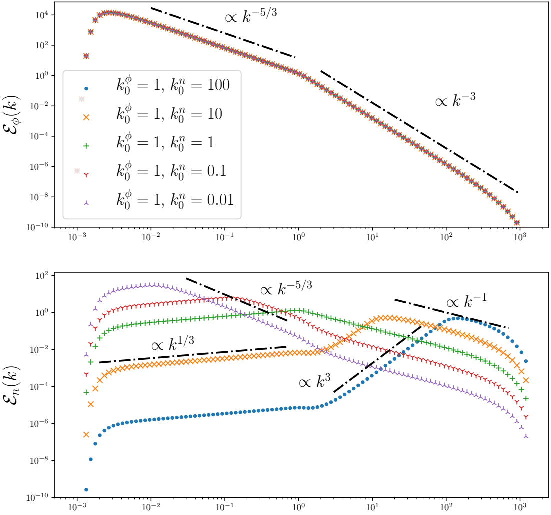
Direct enstrophy cascade
(in the case $n = k^2 \phi$)

Spectral zoo of the passive scalar

Vary injection location:

- ⋆ $\mathcal{I}_\phi$: inject vorticity at $k_0^\phi = 1.0$
- ⋆ $\mathcal{I}_n$: inject passive scalar at $k_0^n = 10^{\{-2\,;\,-1\,;\,0\,;\,1\,;\,2\}}$

$\mathcal{E}_\phi$	$\mathcal{E}_n$	Π_k^ϕ	Π_k^n
$-5/3$	$-11/3$	$-$	0
$-5/3$	$-5/3$	$-$	$+$
$-5/3$	$1/3$	$-$	0
-3	-5	0	0
-3	-1	0	$+$
-3	3	0	0



Shell model coupled to radial profiles[10]

- Transport equations: density $n(r,t)$, pressure $P(r,t)$

$$\begin{aligned} \partial_t n &= \nabla . \left[\left(D_{neo} + D_{turb}\mathcal{E} \right) \nabla n \right] + S_n(r) \\ \partial_t P &= \nabla . \left[\left(\chi_{neo} + \chi_{turb}\mathcal{E} \right) \nabla P \right] + S_Q(r) \end{aligned}$$

- Turbulence evolution: intensity $\mathcal{E}(r,t) = \sum_n \phi_n \phi_n^\star$

$$\begin{aligned} \partial_t \phi_n &= F_n - \nu_L k_n^{-6} \phi_n - \nu_s k_n^4 \phi_n + D_{\mathcal{E}} \nabla^2 \phi_n \\ &\quad - \overline{\alpha} \frac{q k_n \overline{\phi}^\star}{1 + k_n^2} \left[g \left(1 + g^2 k_n^2 - q^2 \right) \phi_{n+1}^\star - \left(1 + \frac{k_n^2}{g^2} - q^2 \right) \phi_{n-1}^\star \right] \\ &\quad + \alpha \frac{k_n^4 \left(g^2 - 1 \right)}{1 + k_n^2} \left[g^{-7} \phi_{n-2}^\star \phi_{n-1}^\star - \left(g^2 + 1 \right) g^{-3} \phi_{n-1}^\star \phi_{n+1}^\star + g^3 \phi_{n+1}^\star \phi_{n+2}^\star \right] \end{aligned}$$

- Mean flow evolution: $\overline{\phi}(r,t)$ at scale q

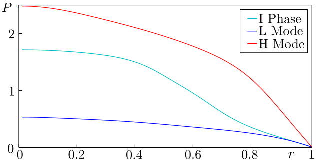
$$\partial_t \overline{\phi} = \overline{\alpha} \sum_n \frac{k_n^3 g \left(g^2 - 1 \right)}{q} \phi_n^\star \phi_{n+1}^\star - \nu_F \left[\overline{\phi} - \phi_{V_E'}(r,t) \right]$$

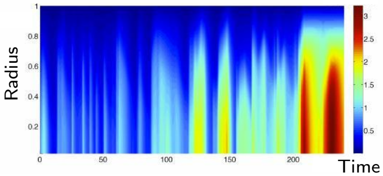
radial force balance: $\phi_{V_E'} = \frac{\eta}{q^2} \nabla P . \nabla n$

[10] V. Berionni, PoP 2017, see also Miki and Diamond for numerous works on similar system

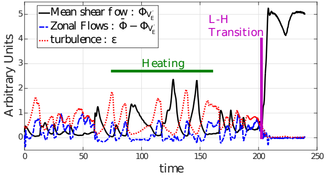
Shell model and LH transition

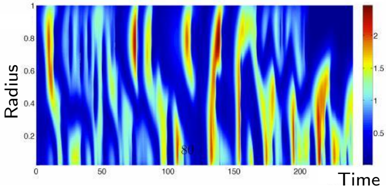
L Mode **H Mode**



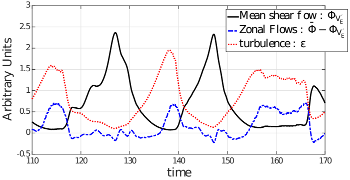


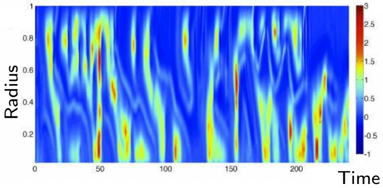
$P(r,t)$





$\mathcal{E}(r,t)$





$\overline{\mathcal{E}}(r,t)$

Back to TEM+TIM system : phase representation

We have to solve :

$$\begin{aligned}
\partial_t f_{s\mathbf{k}} &= ik_\alpha \partial_\psi F_{s0}\, \mathcal{J}_{0s} \phi_{\mathbf{k}} - ik_\alpha \frac{\Omega_d E}{Z_s} f_{s,\mathbf{k}} - \sum_{\mathbf{k}+\mathbf{p}+\mathbf{q}=0} [\mathbf{z} \times \mathbf{p} \,.\, \mathbf{q}]\, \mathcal{J}_{0s} \phi^\star_{\mathbf{p}}\, f^\star_{s\mathbf{q}} \,, \\
C_{\mathbf{k}} \phi_{\mathbf{k}} &= \sum_s Z_s \int_0^{+\infty} \mathcal{J}_{0s} f_{s\mathbf{k}} \sqrt{E}\, dE \,.
\end{aligned}$$

Decompose the distribution function into phase/amplitude: $f_{s\mathbf{k}} = |f_{s\mathbf{k}}|\, e^{i\varphi_{\mathbf{k}}}$

$$\begin{aligned}
\partial_t \,|f_{s\mathbf{k}}| &= -k_\alpha \partial_\psi F_{s0}\, \Im \left[\mathcal{J}_{0s} \phi_{\mathbf{k}} e^{-i\varphi_{\mathbf{k}}} \right] - \sum_{\mathbf{k}+\mathbf{p}+\mathbf{q}} [\mathbf{z} \times \mathbf{p} \,.\, \mathbf{q}]\, \Re \left[\mathcal{J}_{0s} \phi^\star_{\mathbf{p}}\, |f_{s\mathbf{q}}|\, e^{-i\left(\varphi_{\mathbf{q}}+\varphi_{\mathbf{k}}\right)} \right] \,, \\
\partial_t \varphi_{f_s\mathbf{k}} &= k_\alpha \partial_\psi F_{s0} \frac{\Re \left[\mathcal{J}_{0s} \phi_{\mathbf{k}}\, e^{-i\varphi_{\mathbf{k}}} \right]}{|f_{s\mathbf{k}}|} - k_\alpha \frac{\Omega_d E}{Z_s} \\
&\quad - \sum_{\mathbf{k}+\mathbf{p}+\mathbf{q}=0} [\mathbf{z} \times \mathbf{p} \,.\, \mathbf{q}]\, \Im \left[\mathcal{J}_{0s} \phi^\star_{\mathbf{p}}\, e^{-i\left(\varphi_{\mathbf{q}}+\varphi_{\mathbf{k}}\right)} \right] \frac{|f_{s\mathbf{q}}|}{|f_{s\mathbf{k}}|} \,, \\
C_{\mathbf{k}} \phi_{\mathbf{k}} &= \sum_s Z_s \int_0^{+\infty} \mathcal{J}_{0s}\, |f_{s\mathbf{k}}|\; e^{i\varphi_{\mathbf{k}}} \,.
\end{aligned}$$

Phase representation of TEM+TIM

$$\partial_t |f_{s\mathbf{k}}| = -k_\alpha \partial_\psi F_{s0} \Im \left[\mathcal{J}_{0s} \phi_{\mathbf{k}} e^{-i\varphi_{\mathbf{k}}} \right] - \sum_{\mathbf{k}+\mathbf{p}+\mathbf{q}} [\mathbf{z} \times \mathbf{p} \,.\, \mathbf{q}] \Re \left[\mathcal{J}_{0s} \phi^\star_{\mathbf{p}} |f_{s\mathbf{q}}| e^{-i\left(\varphi_{\mathbf{q}}+\varphi_{\mathbf{k}}\right)} \right] ,$$

$$\partial_t \varphi_{\mathbf{k}} = -k_\alpha \frac{\Omega_d E}{Z_s} + k_\alpha \partial_\psi F_{s0} \frac{\Re \left[\mathcal{J}_{0s} \phi_{\mathbf{k}} \, e^{-i\varphi_{\mathbf{k}}} \right]}{|f_{s\mathbf{k}}|} - \sum_{\mathbf{k}+\mathbf{p}+\mathbf{q}=0} [\mathbf{z} \times \mathbf{p} \,.\, \mathbf{q}] \Im \left[\mathcal{J}_{0s} \phi^\star_{\mathbf{p}} \, e^{-i\left(\varphi_{\mathbf{q}}+\varphi_{\mathbf{k}}\right)} \right] \frac{|f_{s\mathbf{q}}|}{|f_{s\mathbf{k}}|} ,$$

$$C_{\mathbf{k}} \phi_{\mathbf{k}} = \sum_s Z_s \int_0^{+\infty} \mathcal{J}_{0s} |f_{s\mathbf{k}}| \; e^{i\varphi_{\mathbf{k}}} \left(= C_k |\phi_k| \, e^{i\overline{\varphi_{\mathbf{k}}}} \right) .$$

⋆ Injection wrt amplitude: linear, due to equilibrium gradients $\propto \partial_\psi F_{s0}$

contains the radial fluxes of heat and particles

anisotropy: $k_\alpha = k \sin\theta_k$

phase relationship: $\Im \left[J_{0s} \phi_{\mathbf{k}} e^{-i\varphi_k} \right] \propto \sin\left(\overline{\varphi_{\mathbf{k}}} - \varphi_{\mathbf{k}} \right)$

$\Rightarrow$ null for $\overline{\varphi_{\mathbf{k}}} - \varphi_{\mathbf{k}} = 0$

correspond to **stable drift waves**

Phase representation of TEM+TIM

$$\partial_t |f_{s\mathbf{k}}| = -k_\alpha \partial_\psi F_{s0} \Im\left[\mathcal{J}_{0s}\phi_{\mathbf{k}} e^{-i\varphi_{\mathbf{k}}}\right] - \sum_{\mathbf{k}+\mathbf{p}+\mathbf{q}} [\mathbf{z}\times\mathbf{p}\,.\,\mathbf{q}]\,\Re\left[\mathcal{J}_{0s}\phi^\star_{\mathbf{p}} |f_{s\mathbf{q}}|\, e^{-i\left(\varphi_{\mathbf{q}}+\varphi_{\mathbf{k}}\right)}\right],$$

$$\partial_t \varphi_{\mathbf{k}} = -k_\alpha \frac{\Omega_d E}{Z_s} + k_\alpha \partial_\psi F_{s0} \frac{\Re\left[\mathcal{J}_{0s}\phi_{\mathbf{k}}\, e^{-i\varphi_{\mathbf{k}}}\right]}{|f_{s\mathbf{k}}|} - \sum_{\mathbf{k}+\mathbf{p}+\mathbf{q}=0} [\mathbf{z}\times\mathbf{p}\,.\,\mathbf{q}]\,\Im\left[\mathcal{J}_{0s}\phi^\star_{\mathbf{p}}\, e^{-i\left(\varphi_{\mathbf{q}}+\varphi_{\mathbf{k}}\right)}\right] \frac{|f_{s\mathbf{q}}|}{|f_{s\mathbf{k}}|},$$

$$C_{\mathbf{k}}\phi_{\mathbf{k}} = \sum_s Z_s \int_0^{+\infty} \mathcal{J}_{0s} |f_{s\mathbf{k}}|\, e^{i\varphi_{\mathbf{k}}} \left(= C_k |\phi_k|\, e^{i\overline{\varphi_{\mathbf{k}}}}\right).$$

- ⋆ Injection wrt amplitude
- ⋆ Nonlinear transfers: triangles $\mathbf{k}+\mathbf{p}+\mathbf{q}=0$

phase relationship: $\Re\left[\mathcal{J}_{0s}\phi^\star_{\mathbf{p}} |f_{s\mathbf{q}}|\, e^{-i\left(\varphi_{\mathbf{q}}+\varphi_{\mathbf{k}}\right)}\right] \propto \cos\left(\varphi_{\mathbf{k}}+\varphi_{\mathbf{q}}+\overline{\varphi_{\mathbf{p}}}\right)$

max for $\varphi_{\mathbf{k}}+\overline{\varphi_{\mathbf{p}}}+\varphi_{\mathbf{q}}=0$

Phase representation of TEM+TIM

$$\partial_t |f_{s\mathbf{k}}| = -k_\alpha \partial_\psi F_{s0} \Im \left[\mathcal{J}_{0s}\phi_{\mathbf{k}} e^{-i\varphi_{\mathbf{k}}}\right] - \sum_{\mathbf{k}+\mathbf{p}+\mathbf{q}} [\mathbf{z}\times\mathbf{p}\,.\,\mathbf{q}] \Re\left[\mathcal{J}_{0s}\phi^\star_{\mathbf{p}} |f_{s\mathbf{q}}| e^{-i\left(\varphi_{\mathbf{q}}+\varphi_{\mathbf{k}}\right)}\right],$$

$$\partial_t \varphi_{\mathbf{k}} = -k_\alpha \frac{\Omega_d E}{Z_s} + k_\alpha \partial_\psi F_{s0} \frac{\Re\left[\mathcal{J}_{0s}\phi_{\mathbf{k}}\, e^{-i\varphi_{\mathbf{k}}}\right]}{|f_{s\mathbf{k}}|} - \sum_{\mathbf{k}+\mathbf{p}+\mathbf{q}=0} [\mathbf{z}\times\mathbf{p}\,.\,\mathbf{q}] \Im\left[\mathcal{J}_{0s}\phi^\star_{\mathbf{p}}\, e^{-i\left(\varphi_{\mathbf{q}}+\varphi_{\mathbf{k}}\right)}\right] \frac{|f_{s\mathbf{q}}|}{|f_{s\mathbf{k}}|},$$

$$C_{\mathbf{k}}\phi_{\mathbf{k}} = \sum_s Z_s \int_0^{+\infty} \mathcal{J}_{0s} |f_{s\mathbf{k}}|\, e^{i\varphi_{\mathbf{k}}} \left(= C_k |\phi_k|\, e^{i\overline{\varphi_{\mathbf{k}}}}\right).$$

- ⋆ Injection wrt amplitude
- ⋆ Nonlinear transfers
- ⋆ ballistic phase:

$\propto Z_s$

linear growth/decrease of phase

$\propto k_\alpha, E$

Phase representation of TEM+TIM

$$
\begin{aligned}
\partial_t |f_{s\mathbf{k}}| &= -k_\alpha \partial_\psi F_{s0} \, \Im \left[\mathcal{J}_{0s} \phi_{\mathbf{k}} e^{-i\varphi_{\mathbf{k}}} \right] - \sum_{\mathbf{k}+\mathbf{p}+\mathbf{q}} [\mathbf{z} \times \mathbf{p} \, . \, \mathbf{q}] \, \Re \left[\mathcal{J}_{0s} \phi_{\mathbf{p}}^\star \, |f_{s\mathbf{q}}| \, e^{-i\left(\varphi_{\mathbf{q}} + \varphi_{\mathbf{k}}\right)} \right], \\
\partial_t \varphi_{\mathbf{k}} &= -k_\alpha \frac{\Omega_d E}{Z_s} + k_\alpha \partial_\psi F_{s0} \frac{\Re \left[\mathcal{J}_{0s} \phi_{\mathbf{k}} \, e^{-i\varphi_{\mathbf{k}}} \right]}{|f_{s\mathbf{k}}|} \\
&\quad - \sum_{\mathbf{k}+\mathbf{p}+\mathbf{q}=0} [\mathbf{z} \times \mathbf{p} \, . \, \mathbf{q}] \, \Im \left[\mathcal{J}_{0s} \phi_{\mathbf{p}}^\star \, e^{-i\left(\varphi_{\mathbf{q}} + \varphi_{\mathbf{k}}\right)} \right] \frac{|f_{s\mathbf{q}}|}{|f_{s\mathbf{k}}|}, \\
C_{\mathbf{k}} \phi_{\mathbf{k}} &= \sum_s Z_s \int_0^{+\infty} \mathcal{J}_{0s} \, |f_{s\mathbf{k}}| \, e^{i\varphi_{\mathbf{k}}} \left(= C_k \, |\phi_k| \, e^{i\overline{\varphi_{\mathbf{k}}}} \right) .
\end{aligned}
$$

- ⋆ Injection wrt amplitude
- ⋆ Nonlinear transfers
- ⋆ ballistic phase
- ⋆ phase coupling wrt energy E:

background gradients $\propto \partial_\psi F_{s0}$
anisotropy: $\propto k_\alpha = k \sin\theta_k$
phase relationship: $\Re \left[J_{0s} \phi_{\mathbf{k}} \, e^{-i\varphi_{\mathbf{k}}} \right] \propto \cos\left(\overline{\varphi_{\mathbf{k}}} - \varphi_{\mathbf{k}} \right)$
$\Rightarrow$ max for $\overline{\varphi_{\mathbf{k}}} - \varphi_{\mathbf{k}} = 0$

Kuramoto model: $\boxed{d_t\theta_i = \omega_i + Kr \sin\left(\psi - \theta_i\right)}$

Phase representation of TEM+TIM

$$
\begin{aligned}
\partial_t |f_{s\mathbf{k}}| &= -k_\alpha \partial_\psi F_{s0}\, \Im\left[\mathcal{J}_{0s}\phi_{\mathbf{k}} e^{-i\varphi_{\mathbf{k}}}\right] - \sum_{\mathbf{k}+\mathbf{p}+\mathbf{q}} [\mathbf{z}\times\mathbf{p}\,.\,\mathbf{q}]\, \Re\left[\mathcal{J}_{0s}\phi^\star_{\mathbf{p}}\, |f_{s\mathbf{q}}|\, e^{-i\left(\varphi_{\mathbf{q}}+\varphi_{\mathbf{k}}\right)}\right], \\
\partial_t \varphi_{\mathbf{k}} &= -k_\alpha \frac{\Omega_d E}{Z_s} + k_\alpha \partial_\psi F_{s0} \frac{\Re\left[\mathcal{J}_{0s}\phi_{\mathbf{k}}\, e^{-i\varphi_{\mathbf{k}}}\right]}{|f_{s\mathbf{k}}|} \\
&\quad - \sum_{\mathbf{k}+\mathbf{p}+\mathbf{q}=0} [\mathbf{z}\times\mathbf{p}\,.\,\mathbf{q}]\, \Im\left[\mathcal{J}_{0s}\phi^\star_{\mathbf{p}}\, e^{-i\left(\varphi_{\mathbf{q}}+\varphi_{\mathbf{k}}\right)}\right] \frac{|f_{s\mathbf{q}}|}{|f_{s\mathbf{k}}|}, \\
C_{\mathbf{k}}\phi_{\mathbf{k}} &= \sum_s Z_s \int_0^{+\infty} \mathcal{J}_{0s}\, |f_{s\mathbf{k}}|\, e^{i\varphi_{\mathbf{k}}} \left(= C_k\, |\phi_k|\, e^{i\overline{\varphi_{\mathbf{k}}}}\right).
\end{aligned}
$$

- ⋆ Injection wrt amplitude
- ⋆ Nonlinear transfers
- ⋆ ballistic phase
- ⋆ phase coupling wrt energy E
- ⋆ phase coupling wrt k:

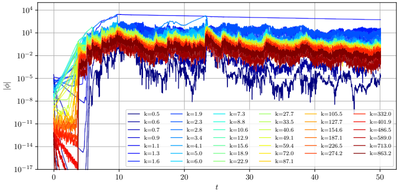
triangles $\mathbf{k}+\mathbf{p}+\mathbf{q}=0$

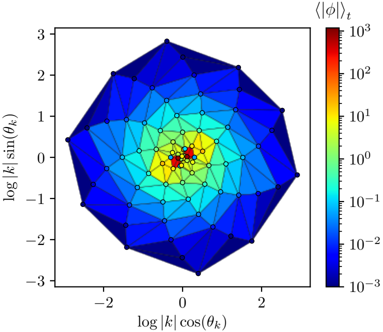
$$\Im\left[\mathcal{J}_{0s}\phi^\star_{\mathbf{p}} e^{-i\left(\varphi_{\mathbf{q}}+\varphi_{\mathbf{k}}\right)}\right] \propto \sin\left(\varphi_{\mathbf{k}} + \overline{\varphi_{\mathbf{p}}} + \varphi_{\mathbf{q}}\right)$$

$\Rightarrow$ minimal for maximal nonlinear transfers

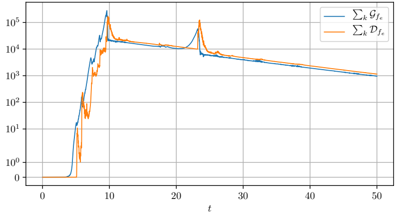
Phase dynamics in TEM

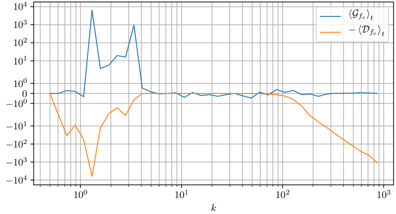
TEM driven simulation + phase representation



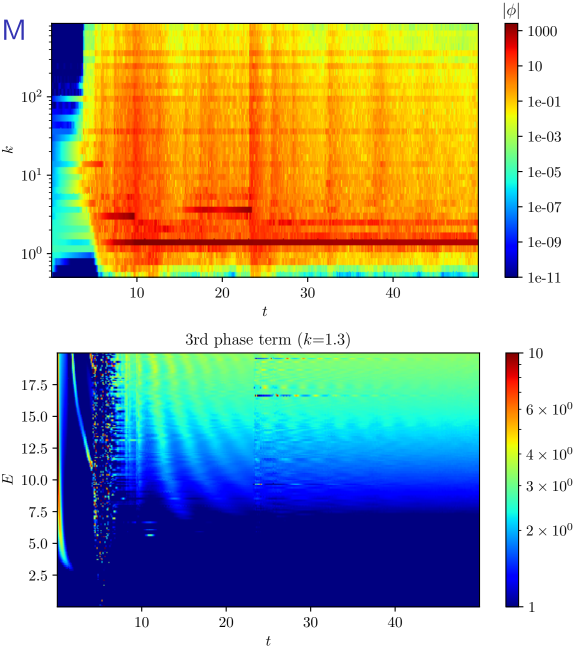


Phase dynamics in TEM





Phase dynamics in TEM



Phase dynamics in TEM