Multiscale Asymptotic Paraxial Models for Approximating Vlasov-Maxwell Equations

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Outline

- Introduction
- Governing equations
- Asymptotic expansion
- Paraxial model
- Conclusion

Introduction

- Framework - numerical model of the Vlasov-Maxwell equations for a beam of particles in an axial frame.
- Charged particle beams examples :
  - Particle accelerators (CERN, Gyrotron, etc.)
  - Industrial uses (electron-ion beams for welding and micromachinning, lithography, etc.)
- Full set of equations is coupled and so computationally expensive.
- Simpler model - the paraxial model developed by G. Laval *et al.* for an *ultra-relativistic* beam.
- Purpose of this work - extending the model to lower velocities, and even non-relativistic cases (multiscale model).

Vlasov-Maxwell equations (1)

- Vlasov equation

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla_{\mathbf{x}} f + \mathbf{F} \cdot \nabla_{\mathbf{p}} f = 0$$

- The force is the Lorentz force

$$\mathbf{F} = q \left(\mathbf{E} + \mathbf{v} \times \mathbf{B} \right)$$

Vlasov-Maxwell equations (2)

- Maxwell equations

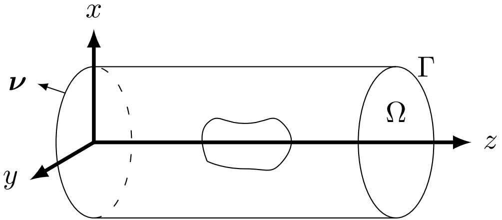
$$\begin{cases} \operatorname{div} \mathbf{E} & = & \dfrac{\rho}{\varepsilon_0} \\ \operatorname{div} \mathbf{B} & = & 0 \\ \mathbf{curl} \ \mathbf{E} & = & -\dfrac{\partial \mathbf{B}}{\partial t} \\ \mathbf{curl} \ \mathbf{B} & = & \mu_0 \left(\mathbf{J} + \varepsilon_0 \dfrac{\partial \mathbf{E}}{\partial t} \right) \end{cases}$$

- Charge and current densities

$$\begin{cases} \rho & = & q \int\limits_{\mathbb{R}^3} f \, \mathrm{d}p \\ \mathbf{J} & = & q \int\limits_{\mathbb{R}^3} \mathbf{v} f \, \mathrm{d}p \end{cases}$$

Beam frame

- Specific geometry



- Change of Variables (to a frame moving along the z axis with a fraction of the speed of light βc)

$$\zeta = \beta ct - z, \quad v_\zeta = \beta c - v_z \quad (0 < \beta < 1)$$

- Derivatives take the form

$$\left(\frac{\partial}{\partial z}, \frac{\partial}{\partial v_z}, \frac{\partial}{\partial t}\right) \to \left(-\frac{\partial}{\partial \zeta}, -\frac{\partial}{\partial v_\zeta}, \frac{\partial}{\partial t} + \beta c \frac{\partial}{\partial \zeta}\right)$$

Scaling

- Some assumptions
  1. The characteristic beam length l is much smaller than the length of the device.
  2. The characteristic transverse velocity $\bar{v}$ is much smaller than the speed of light c.
- Define

$$\eta \equiv \frac{\bar{v}}{c} \ll 1$$

- Also define characteristic time $T = \dfrac{l}{\bar{v}}$ and characteristic longitudinal velocity $\bar{w}$.

Transverse and longitudinal quantities

- It will be convenient to define the transverse quantities

$$\mathbf{x}_\perp = (x, y) \qquad \mathbf{v}_\perp = (v_x, v_y)$$
$$\mathbf{grad}_\perp \phi = \left(\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}\right) \qquad \mathbf{curl}_\perp \phi = \left(\frac{\partial \phi}{\partial y}, -\frac{\partial \phi}{\partial x}\right)$$
$$\mathrm{div}_\perp \mathbf{A}_\perp = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} \quad \mathrm{curl}_\perp \mathbf{A}_\perp = \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}$$

- Also define the *Pseudo-field*

$$\mathcal{E}_\perp = (E_x - \beta c B_y, E_y + \beta c B_x)$$

Dimensionless relativistic Vlasov equation (1)

Applying these changes to the Vlasov equation

$$\frac{\partial f}{\partial t} + \mathbf{v}_\perp \cdot \mathbf{grad}_\perp f + \frac{\bar{w}}{\bar{v}} v_\zeta \frac{\partial f}{\partial \zeta}$$
$$+ \operatorname{div}_{\mathbf{v}_\perp} \left(\frac{1}{\gamma} \left(\left(\mathbf{I} - \eta^2 \mathbf{v}_\perp \otimes \mathbf{v}_\perp \right) \cdot \mathbf{F}_\perp - \eta \left(\beta - \frac{\bar{w}}{c} v_\zeta \right) \mathbf{v}_\perp F_z \right) f \right)$$
$$+ \frac{\bar{v}}{\bar{w}} \frac{\partial}{\partial v_\zeta} \left(\frac{1}{\gamma} \left(\eta \mathbf{v}_\perp \cdot \mathbf{F}_\perp \left(\beta - \frac{\bar{w}}{c} v_\zeta \right) \right.\right.$$
$$\left.\left. + \left(\left(\frac{\bar{w}}{c} \right)^2 {v_\zeta}^2 - 2\beta \frac{\bar{w}}{c} v_\zeta + \beta^2 - 1 \right) F_z \right) f \right) = 0$$

with

$$\gamma = \left(1 - \beta^2 + 2\beta \frac{\bar{w}}{c} v_\zeta - \eta^2 \left| \mathbf{v}_\perp \right|^2 - \left(\frac{\bar{w}}{c} \right)^2 {v_\zeta}^2 \right)^{-\frac{1}{2}}$$

Dimensionless relativistic Vlasov equation (2)

- With boundary conditions

$$f = 0 \quad \text{for} \quad \begin{cases} (\mathbf{x}_\perp, \zeta) \in \Gamma \times (0, Z) & \mathbf{v} \cdot \boldsymbol{\nu} < 0 \\ \mathbf{x}_\perp \in \Omega, \zeta = 0 & v_\zeta > 0 \\ \mathbf{x}_\perp \in \Omega, \zeta = Z & v_\zeta < 0 \end{cases}$$

- And the force takes the form

$$\begin{aligned} \mathbf{F}_\perp &= \mathcal{E}_\perp + \left(\eta B_z \mathbf{v}_\perp + \frac{\bar{w}}{c} v_\zeta \mathbf{B}_\perp \right) \times \hat{\mathbf{e}}_z \\ F_z &= E_z + \eta \left(v_x B_y - v_y B_x \right) \end{aligned}$$

Dimensionless Maxwell equations (1)

$$\eta\frac{\partial \mathbf{E}_\perp}{\partial t} + \frac{1}{\beta}\frac{\partial}{\partial \zeta}\left(\mathcal{E}_\perp - \left(1-\beta^2\right)\mathbf{E}_\perp\right) - \mathbf{curl}_\perp B_z = -\eta \mathbf{J}_\perp$$

$$\eta\frac{\partial E_z}{\partial t} + \frac{1}{\beta}\mathrm{div}_\perp\left(\mathcal{E}_\perp - \left(1-\beta^2\right)\mathbf{E}_\perp\right) = \frac{\bar{w}}{c}J'_\zeta$$

$$\mathrm{div}_\perp \mathbf{E}_\perp - \frac{\partial E_z}{\partial \zeta} = \rho$$

$$\eta\frac{\partial \mathbf{B}_\perp}{\partial t} + \frac{\partial}{\partial \zeta}\left(\mathcal{E}_\perp \times \hat{\mathbf{e}}_z\right) + \mathbf{curl}_\perp E_z = \mathbf{0}$$

$$\eta\frac{\partial B_z}{\partial t} + \mathrm{curl}_\perp \mathcal{E}_\perp = 0$$

$$\mathrm{div}_\perp \mathbf{B}_\perp - \frac{\partial B_z}{\partial \zeta} = 0$$

Dimensionless Maxwell equations (2)

With boundary conditions

$$\mathbf{E}_\perp \cdot \boldsymbol{\tau} = 0$$

$$E_z = 0$$

$$\mathcal{E}_\perp \cdot \boldsymbol{\tau} = \beta \mathbf{B}_\perp \cdot \boldsymbol{\nu}$$

$$\left(\eta \frac{\partial}{\partial t} + \beta \frac{\partial}{\partial \zeta}\right)(\mathbf{B}_\perp \cdot \boldsymbol{\nu}) = 0$$

$$\eta \iint\limits_{\Omega} \frac{\partial B_z}{\partial t}\, \mathrm{d}\mathbf{x}_\perp + \beta \oint\limits_{\Gamma} \mathbf{B}_\perp \cdot \boldsymbol{\nu}\, \mathrm{d}l = 0$$

$$\iint\limits_{\Omega} \left(\eta \frac{\partial}{\partial t} + \beta \frac{\partial}{\partial \zeta}\right) B_z\, \mathrm{d}\mathbf{x}_\perp = 0$$

Asymptotic expansion procedure

- Assume that $\bar{w}$ takes the form

$$\bar{w} = a\eta\,(\beta c) + b\eta^2\,(\beta c)$$

  - For $a = 0$, $\bar{w} \sim \eta^2$ i.e. $v_\zeta \ll \mathbf{v}_\perp \ll v_z \approx \beta c$ - *faster case.*
  - For $b = 0$, $\bar{w} \sim \eta$ i.e. $v_\zeta \approx \mathbf{v}_\perp \ll v_z \approx \beta c$ - *slower case.*

- Expand all quantities in powers of η, i.e.

$$f = f^0 + f^1\eta^1 + f^2\eta^2 + \ldots = \sum_{i=0}^{n} f^i\eta^i$$

- Write out the equations for each power of η.

n-th order dimensionless Vlasov equation (1)

$$\frac{\partial f^n}{\partial t} + \mathbf{v}_\perp \cdot \mathbf{grad}_\perp f^n + \beta v_\zeta \frac{\partial}{\partial \zeta}\left(a f^n + b f^{n-1}\right)$$
$$+ \operatorname{div}_{\mathbf{v}_\perp} \sum_{i=0}^{n} \sum_{j=0}^{i} \delta^{n-i} \boldsymbol{\lambda}^{i-j} f^j + \frac{\partial}{\partial v_\zeta} \sum_{i=0}^{n} \sum_{j=0}^{i} \kappa^{n-i} \mu^{i-j} f^j = 0$$

where

$$\delta = \frac{1}{\gamma}, \quad \kappa = \frac{1}{\gamma}\frac{\bar{v}}{\bar{w}}$$

n-th order dimensionless Vlasov equation (2)

$\boldsymbol{\lambda}$ and μ are defined as

$$\boldsymbol{\lambda}^n = \mathbf{F}_\perp^n - \mathbf{v}_\perp \left(\mathbf{v}_\perp \cdot \mathbf{F}_\perp^{n-2}\right) - \beta \mathbf{v}_\perp \left(F_z^{n-1} - v_\zeta \left(a F_z^{n-2} + b F_z^{n-3}\right)\right)$$

$$\begin{aligned}\mu^n = \beta \mathbf{v}_\perp \cdot \left(\mathbf{F}_\perp^{n-1} - v_\zeta \left(a \mathbf{F}_\perp^{n-2} + b \mathbf{F}_\perp^{n-3}\right)\right) \\ - \left(1 - \beta^2\right) F_z^n - 2\beta^2 v_\zeta \left(a F_z^{n-1} + b F_z^{n-2}\right) \\ + \beta^2 v_\zeta^2 \left(a^2 F_z^{n-2} + 2ab F_z^{n-3} + b^2 F_z^{n-4}\right)\end{aligned}$$

n-th order dimensionless Maxwell equations (1)

After some manipulations the equations are separated and take the form

$$\begin{cases} {\nabla_\perp}^2 E_z^n + \left(1-\beta^2\right)\dfrac{\partial^2 E_z^n}{\partial \zeta^2} = \dfrac{\partial}{\partial t}\left(\beta \dfrac{\partial E_z^{n-1}}{\partial \zeta} + \mathrm{curl}_\perp \mathbf{B}_\perp^{n-1}\right) & \\ -\dfrac{\partial}{\partial \zeta}\left(\beta^2\left(aJ_\zeta^{n-1} + bJ_\zeta^{n-2}\right) + \left(1-\beta^2\right)\rho^n\right) & \text{in } \Omega \\ E_z^n = 0 & \text{on } \Gamma \end{cases}$$

n-th order dimensionless Maxwell equations (2)

$$
\left\{
\begin{aligned}
&\mathrm{curl}_\perp \mathcal{E}_\perp^n = -\frac{\partial B_z^{n-1}}{\partial t} \\
&\mathrm{div}_\perp \mathcal{E}_\perp^n = \left(1-\beta^2\right)\left(\frac{\partial E_z^n}{\partial \zeta} + \rho^n\right) \qquad\qquad \text{in } \Omega \\
&\qquad\qquad + \beta^2 \left(a J_\zeta^{n-1} + b J_\zeta^{n-2}\right) - \beta \frac{\partial E_z^{n-1}}{\partial t} \\
&\oint_\Gamma \mathcal{E}_\perp^n \cdot \boldsymbol{\tau}\, \mathrm{d}l = -\iint_\Omega \frac{\partial B_z^{n-1}}{\partial t}\, \mathrm{d}\mathbf{x}_\perp
\end{aligned}
\right.
$$

n-th order dimensionless Maxwell equations (3)

$$
\begin{cases}
\mathbf{curl}_\perp\left(\mathrm{curl}_\perp \mathbf{E}_\perp^n\right) - \left(1-\beta^2\right)\dfrac{\partial^2 \mathbf{E}_\perp^n}{\partial \zeta^2} & \\
= -\dfrac{\partial^2 \mathcal{E}_\perp^n}{\partial \zeta^2} - \mathbf{curl}_\perp\left(\dfrac{\partial B_z^{n-1}}{\partial t}\right) - \beta \dfrac{\partial}{\partial \zeta}\left(\dfrac{\partial \mathbf{E}_\perp^{n-1}}{\partial t} + \mathbf{J}_\perp^{n-1}\right) & \text{in } \Omega \\
\mathbf{E}_\perp^n \cdot \boldsymbol{\tau} = 0 & \text{on } \Gamma
\end{cases}
$$

n-th order dimensionless Maxwell equations (4)

$$\begin{cases} \mathrm{curl}_\perp \mathbf{B}_\perp^n = \dfrac{\partial E_z^{n-1}}{\partial t} + \beta \mathrm{div}_\perp \mathbf{E}_\perp^n - \beta \left(a J_\zeta^{n-1} + b J_\zeta^{n-2} \right) \\ \mathrm{div}_\perp \mathbf{B}_\perp^n = -\dfrac{1}{\beta} \left(\mathrm{curl}_\perp \mathbf{E}_\perp^n + \dfrac{\partial B_z^{n-1}}{\partial t} \right) \\ \oint\limits_\Gamma \mathbf{B}_\perp^n \cdot \boldsymbol{\nu} \, \mathrm{d}l = -\dfrac{1}{\beta} \iint\limits_\Omega \dfrac{\partial B_z^{n-1}}{\partial t} \, \mathrm{d}\mathbf{x}_\perp \end{cases} \quad \text{in } \Omega$$

$$\begin{cases} \dfrac{\partial B_z^n}{\partial \zeta} = \mathrm{div}_\perp \mathbf{B}_\perp^n & \text{in } \Omega \\ \iint\limits_\Omega \dfrac{\partial B_z^n}{\partial \zeta} \, \mathrm{d}\mathbf{x}_\perp = -\dfrac{1}{\beta} \iint\limits_\Omega \dfrac{\partial B_z^{n-1}}{\partial t} \, \mathrm{d}\mathbf{x}_\perp \end{cases}$$

Paraxial model

In order to determine f

1. Start with charge and current densities ρ and $\mathbf{J}$ (from the Vlasov equation).
2. Select relevant case ($a = 0$ or $b = 0$).
3. Solve Maxwell's equations up to order n in the order which they were presented.
4. Calculate the force up to order n.
5. Solve the Vlasov equation up to order n.

Conclusion

- A new paraxial model was presented to solve the Vlasov-Maxwell equations.
- A multiscale extension of an existing paraxial model to not necessarily ultra-relativistic cases.
- The model is hierarchical and closed for each order.
- Quasi static model - all time derivatives are on the LHS.
- Future prospects
  - Numerical simulations
  - Expanding further with a non-constant velocity

Thank you for your time !