Resonant Relaxation of Stars around a supermassive BH

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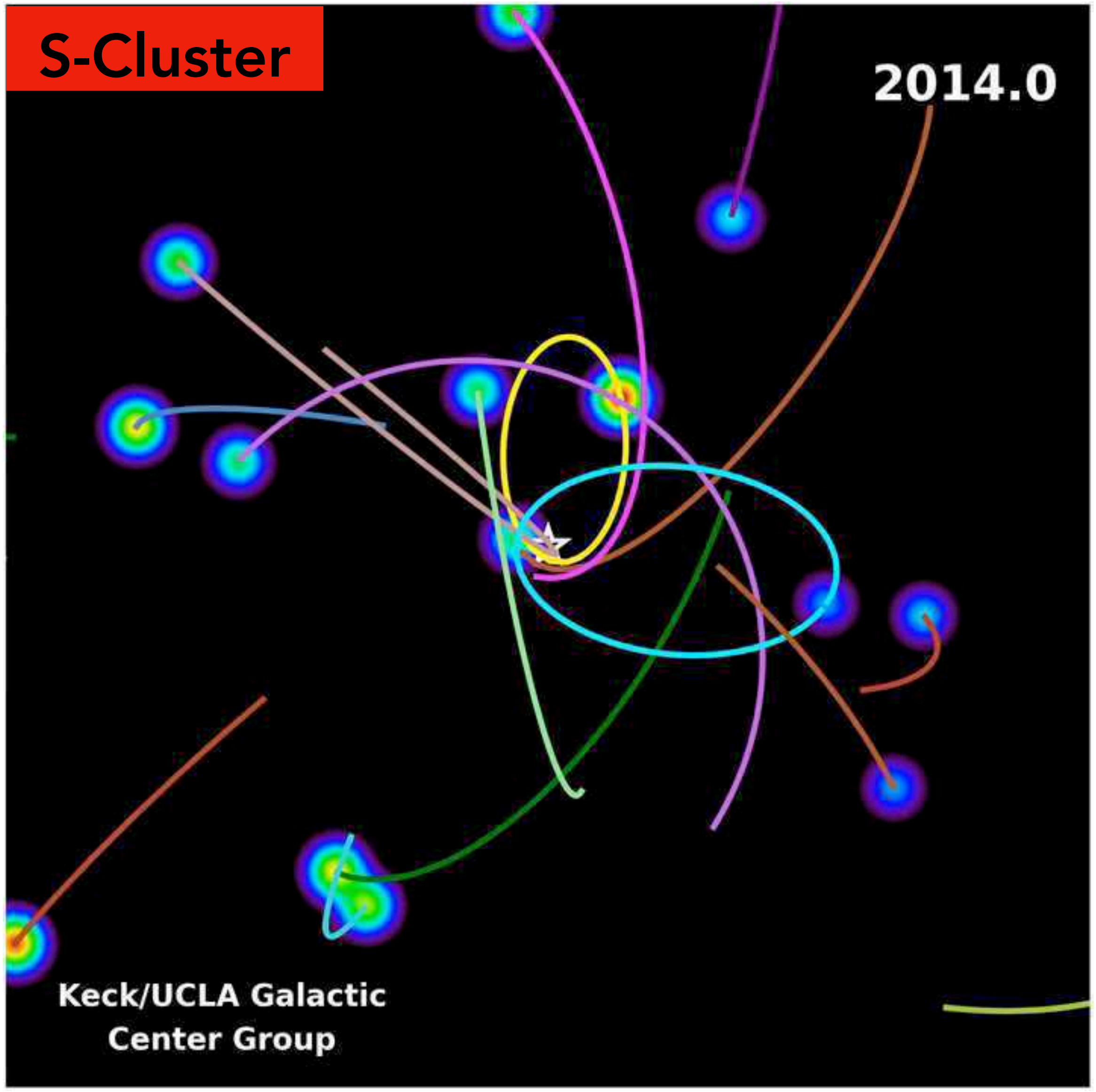
In collaboration with B. Bar-Or, P.-H. Chavanis

The case of galactic centers

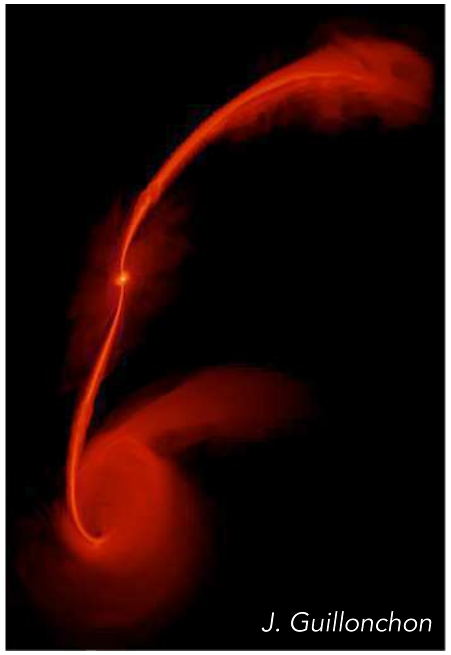


*S-Cluster of **SgrA****
***Densest** stellar system of the galaxy*
*Dynamics dominated by the **central black hole***

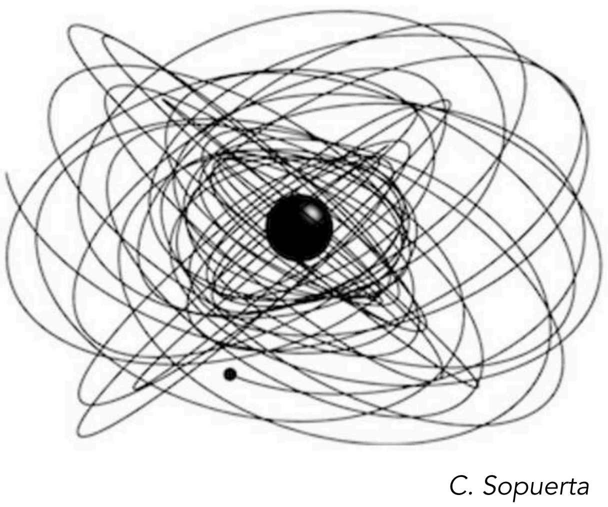
What is the diet of a **supermassive black hole**?

Stellar diffusion in galactic centers
- + ***Origin and structure** of SgrA**
- + *Relaxation in **eccentricity**, **orientation***

Sources of **gravitational waves**
- + *BHs-binary mergers*
- + *TDE, EMRIs*



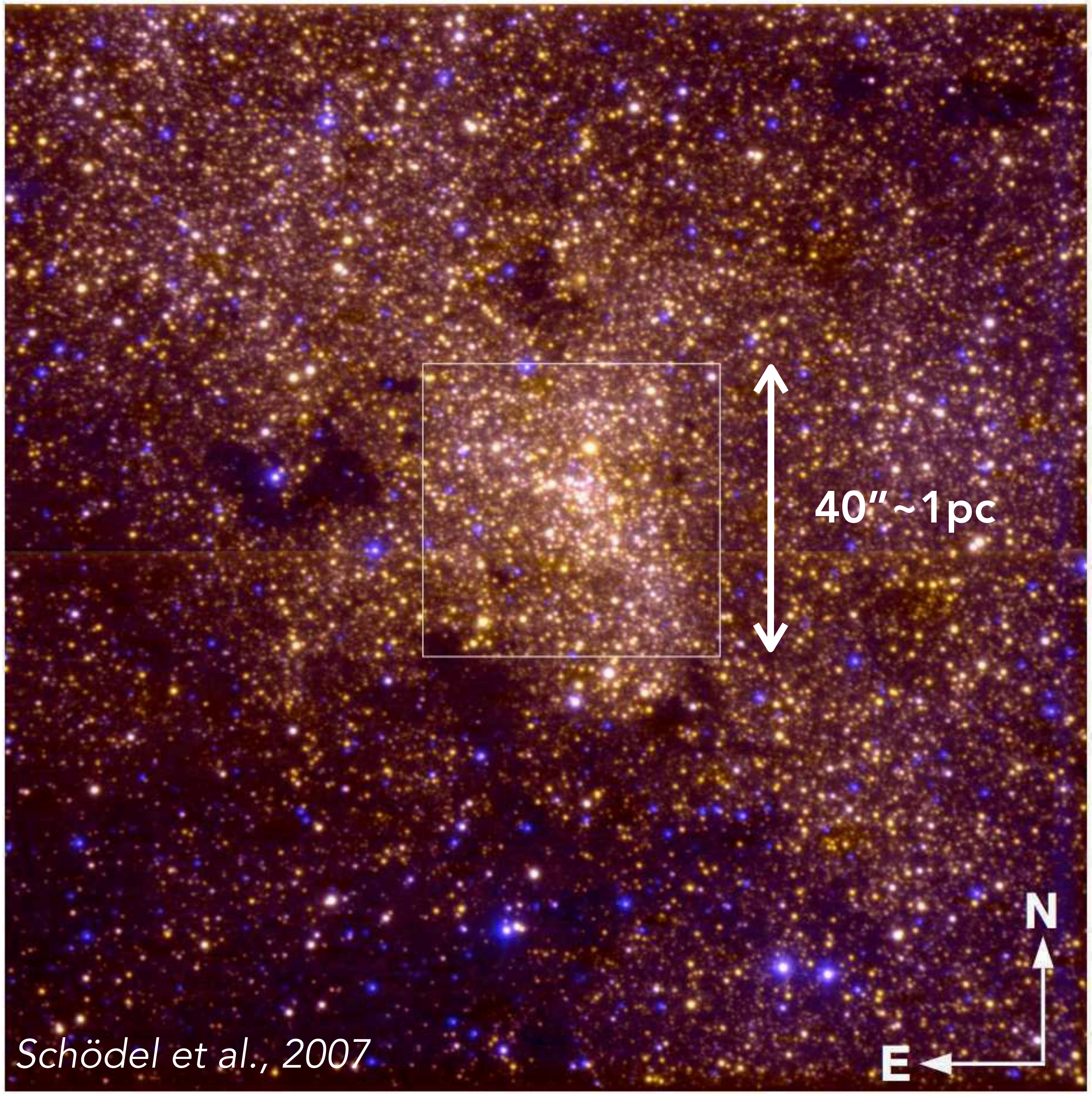
Tidal Disruption Event

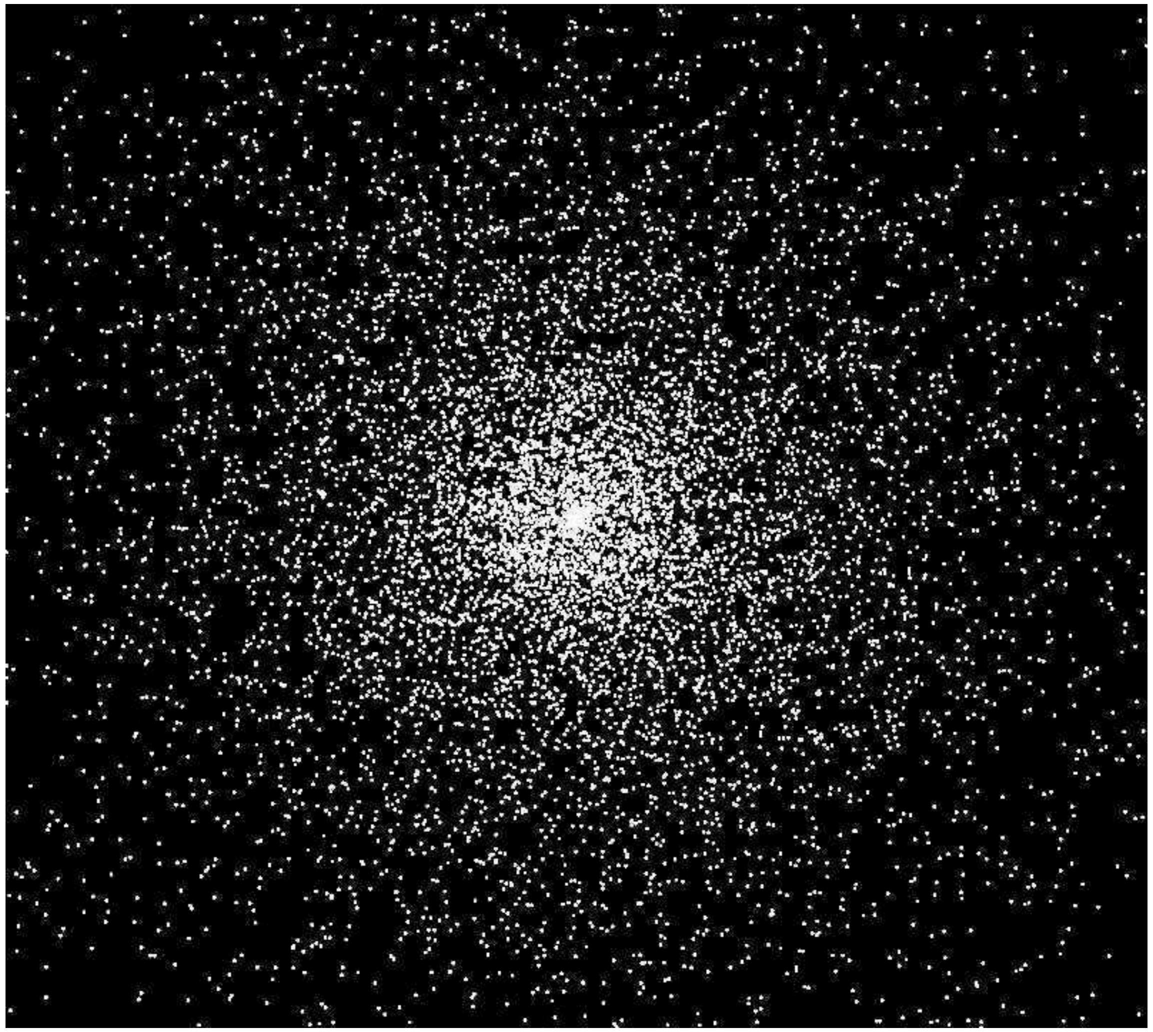


Extreme Mass Ratio Inspiral

What is the long-term dynamics of stars in these very dense systems?

Galactic centers are extremely dense





VLT observations

N-body simulations *(B. Bar-Or)*

Perfect “lab” to investigate the **statistical physics** of a stellar system

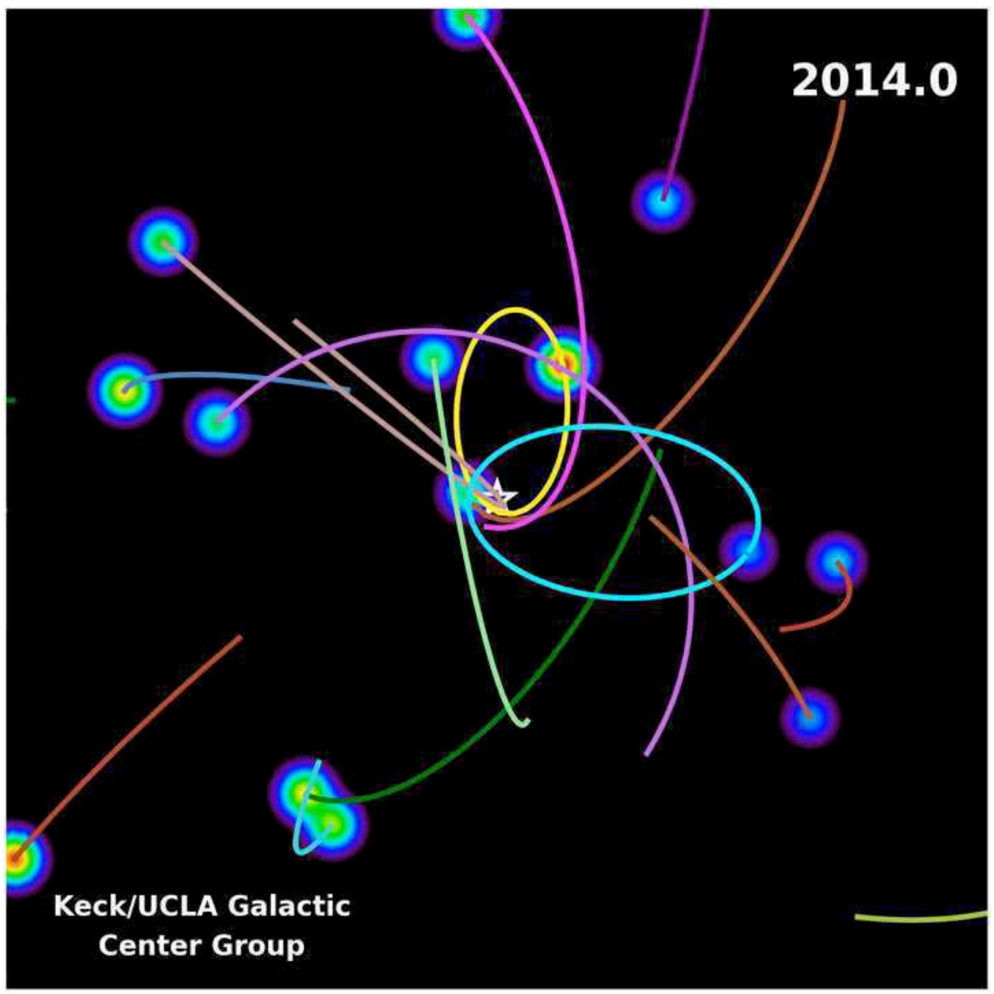
Galactic centers are degenerate

Potential dominated by the SMBH:

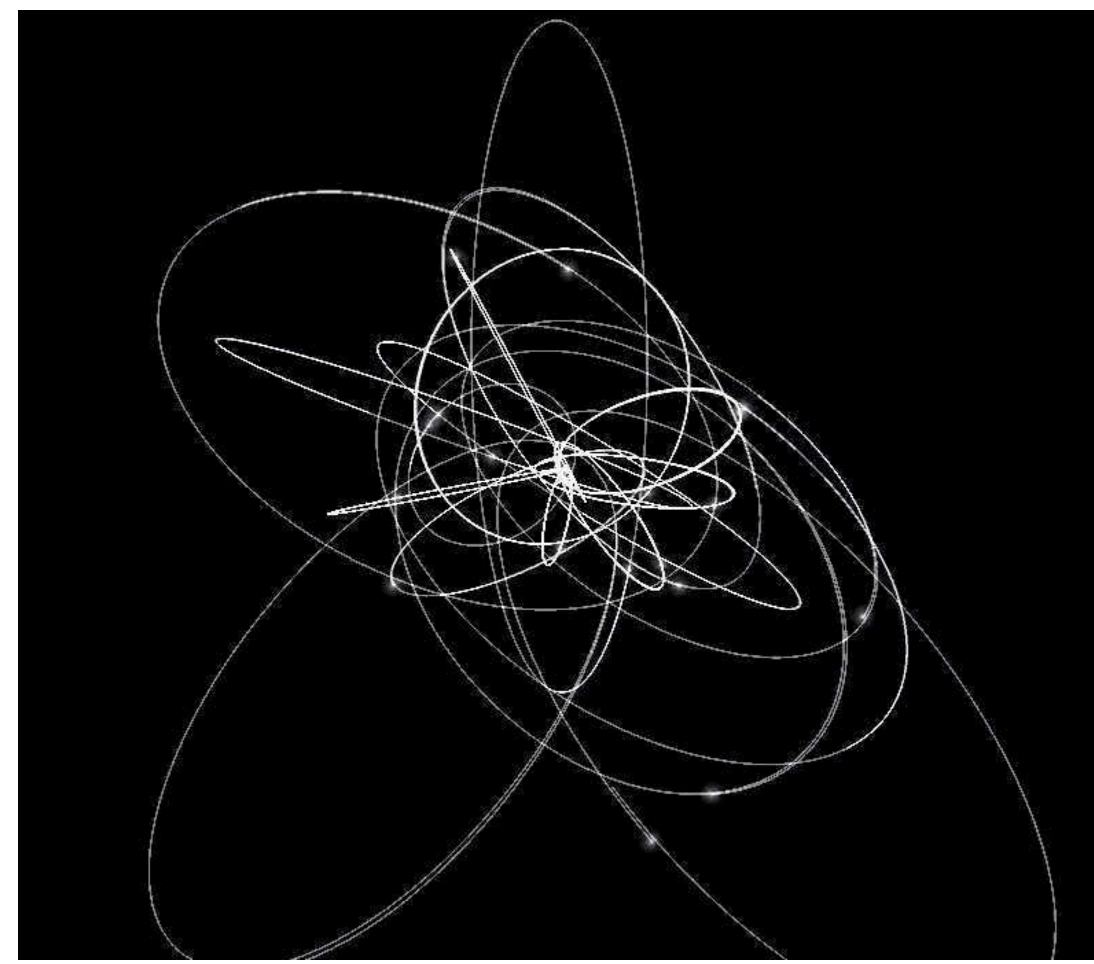
+ Keplerian orbits are **closed**

$$\varepsilon = M_\star / M_\bullet \ll 1$$

Dynamical degeneracy: $\forall \mathbf{J}\,,\ \mathbf{n}\cdot\mathbf{\Omega}_{\mathrm{Kep}}(\mathbf{J}) = 0$



KECK observations

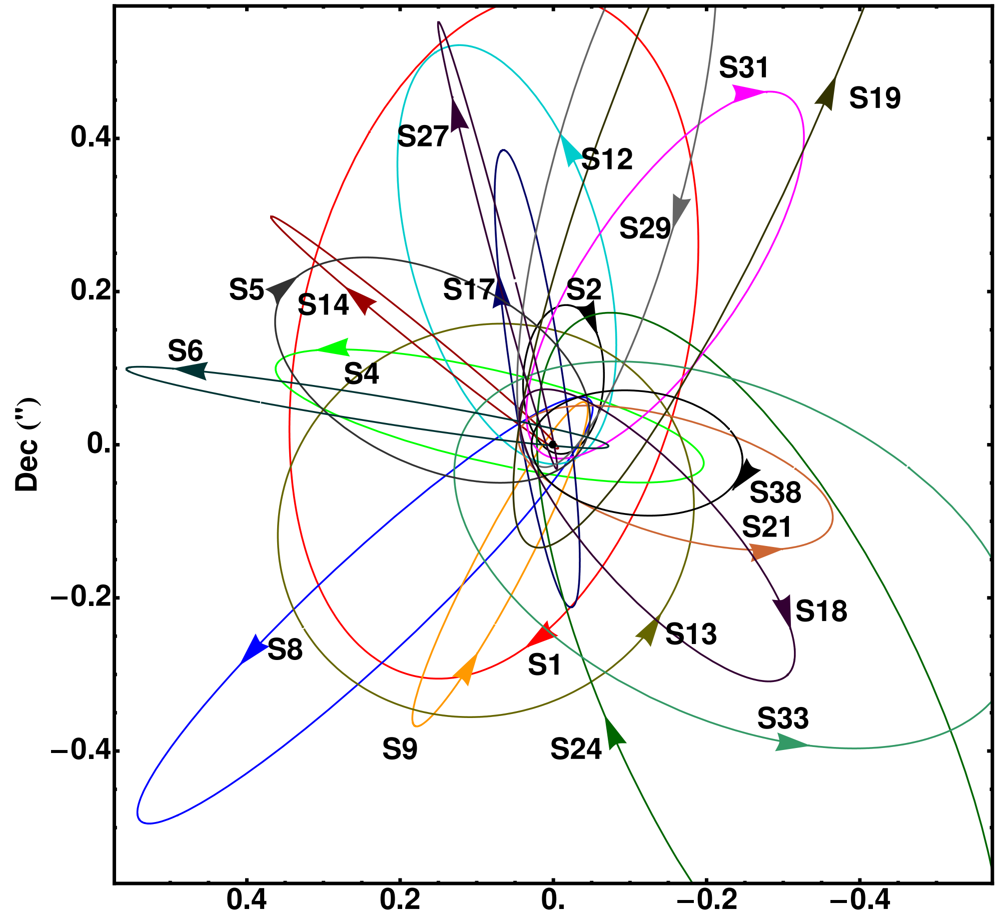


N-body simulations *(B. Bar-Or)*

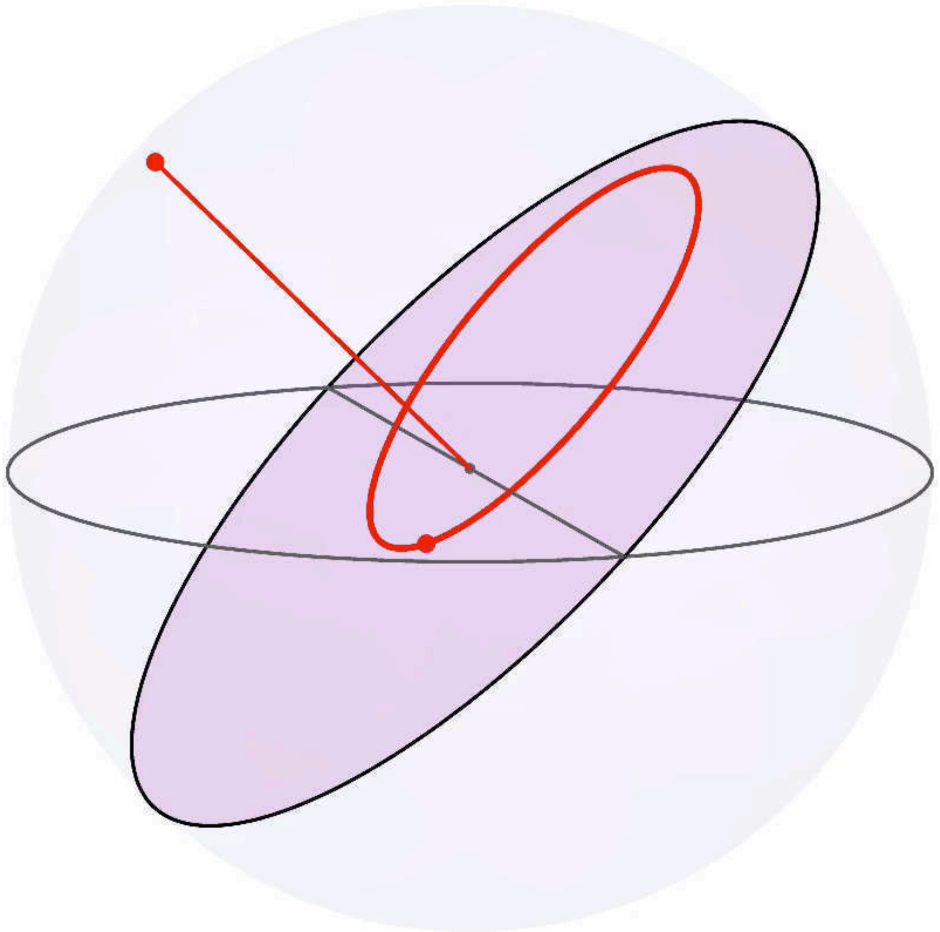
Orbit-average: stars are replaced by Keplerian wires

Describing Keplerian wires

Natural **Angle-Action** coordinates: **Delaunay Variables, i.e. orbital elements**



Gillessen et al., 2009



Degenerate coordinates

$$\mathbf{J} = (I \,,\, L \,,\, L_z)$$
$$\boldsymbol{\theta} = (M \,,\, \omega \,,\, \Omega)$$
$$\boldsymbol{\Omega} = (\Omega_{\mathrm{Kep}} \,,\, 0 \,,\, 0)$$

Wires described by five numbers

+ **Shape**	(a, e)
+ **Phase**	ω
+ **Orientation**	$\hat{\mathbf{L}}$

Wires dynamics

Orbit Average

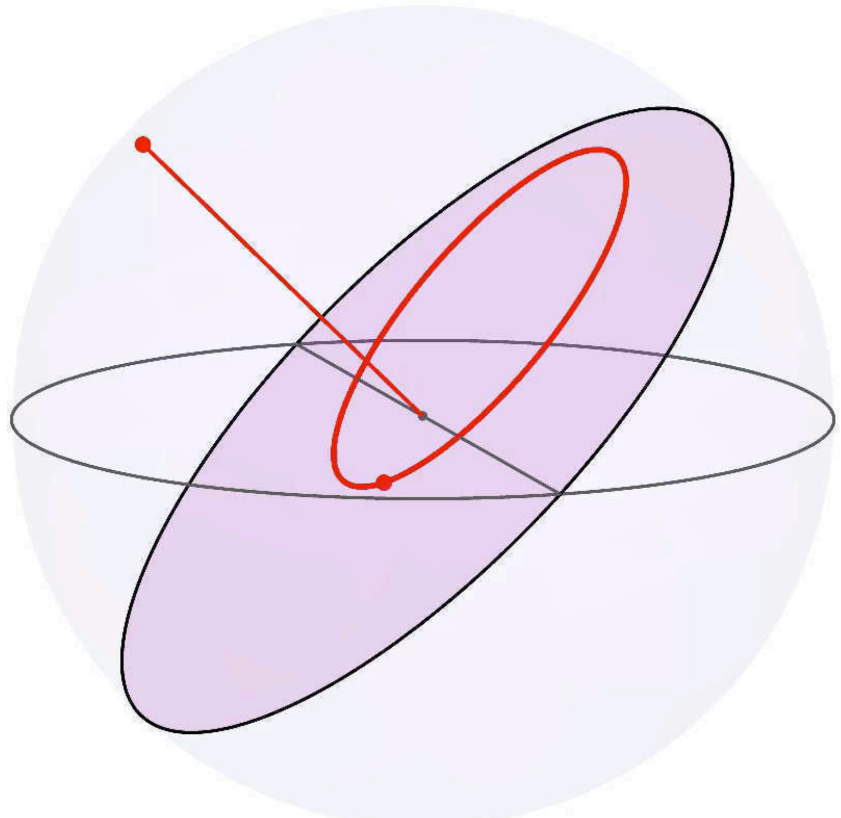
$$J_{\mathrm{fast}} = I(a)$$ **adiabatically conserved**

Wires may **precess constructively**:

+ In-plane precessions

- Spherical cluster mass
- 1PN relativistic **Schwarzschild precession**

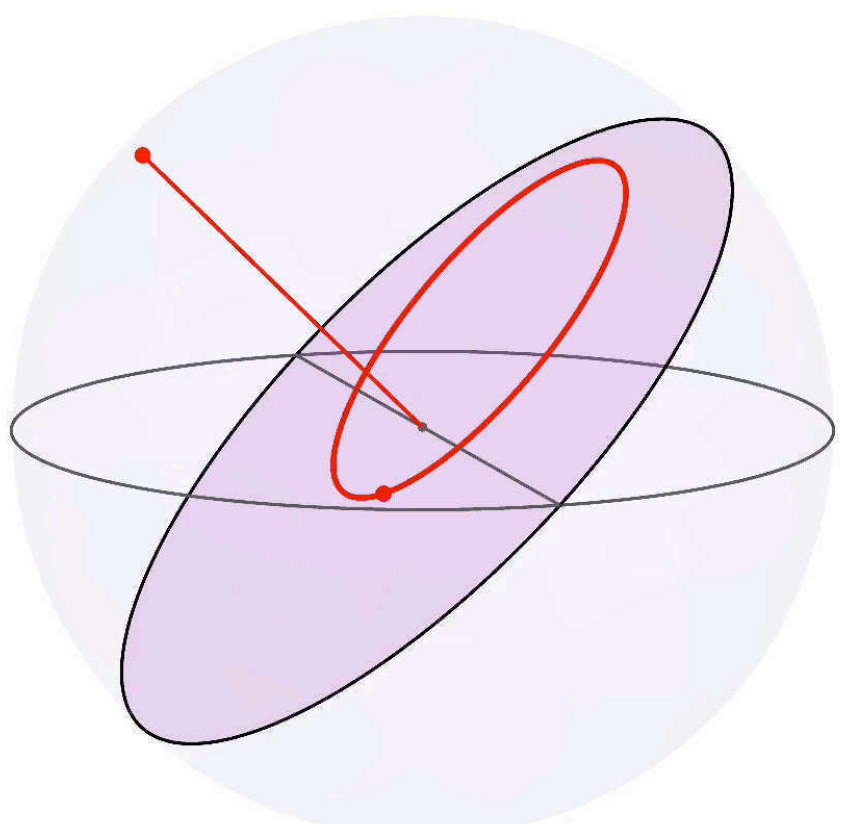
$$\dot{\omega} = \Omega_{\mathrm{prec}} \;\; ; \quad \hat{\mathbf{L}} = \mathrm{cst}\,.$$



+ Out-of-plane precessions

- Triaxial cluster mass
- 1.5PN relativistic **Lense-Thirring precession**

$$\dot{\hat{\mathbf{L}}} = \boldsymbol{\Omega}_{\mathrm{prec}} \;\; ; \quad L = \mathrm{cst}\,.$$



Wires may also **jitter stochastically**

- Finite-N effects $\dot{\hat{\mathbf{L}}} = \eta(t)$

Long-term dynamics of wires

In-plane precessions (L, ω)

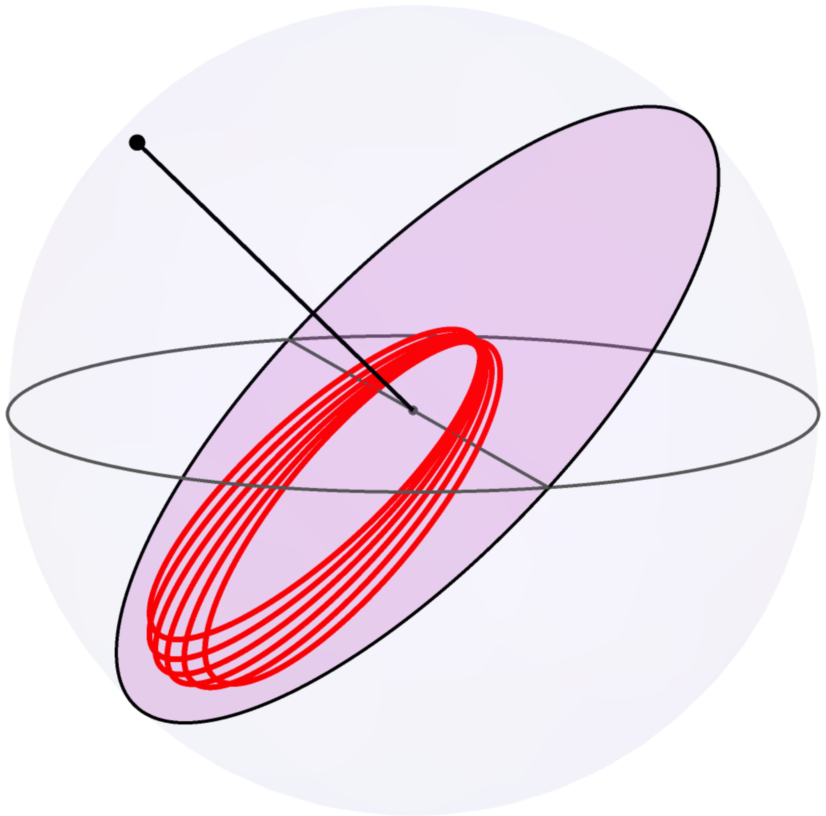
Constructive mean field motion

$$\Omega^{\mathrm{prec}} = \Omega^{\mathrm{prec}}_{\mathrm{self}} + \Omega^{\mathrm{prec}}_{\mathrm{rel}} + \Omega^{\mathrm{prec}}_{\mathrm{ext}}$$

Long-term diffusion of $L = L(e)$

Scalar Resonant Relaxation

See Hamilton's talk



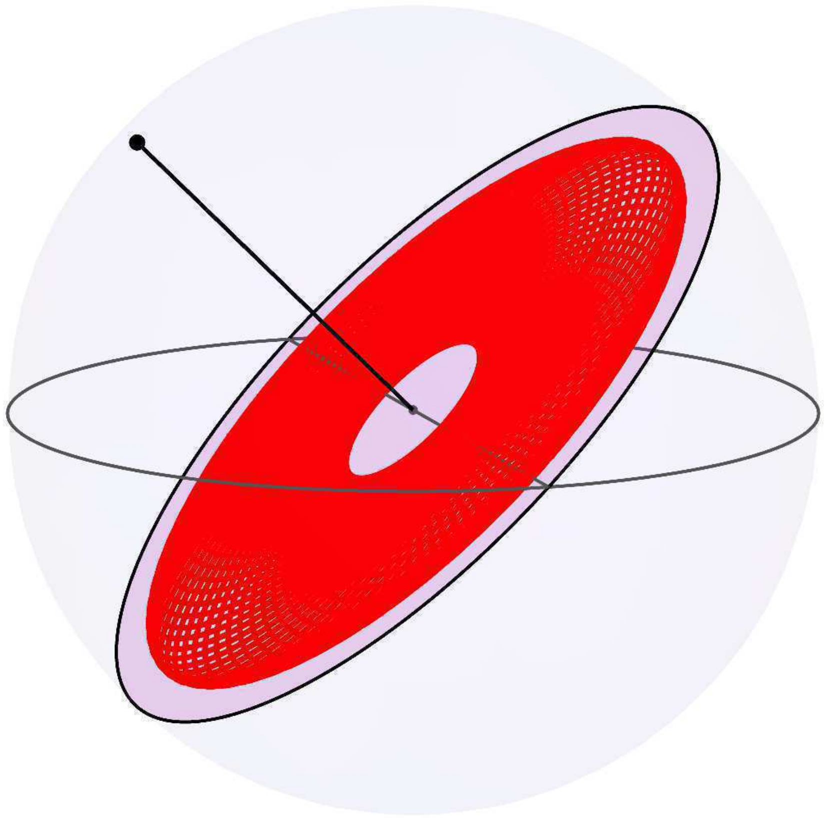
Out-of-plane precessions $\hat{\mathbf{L}}$

No mean field motion

$$\langle \Omega^{\mathrm{prec}} \rangle = 0$$

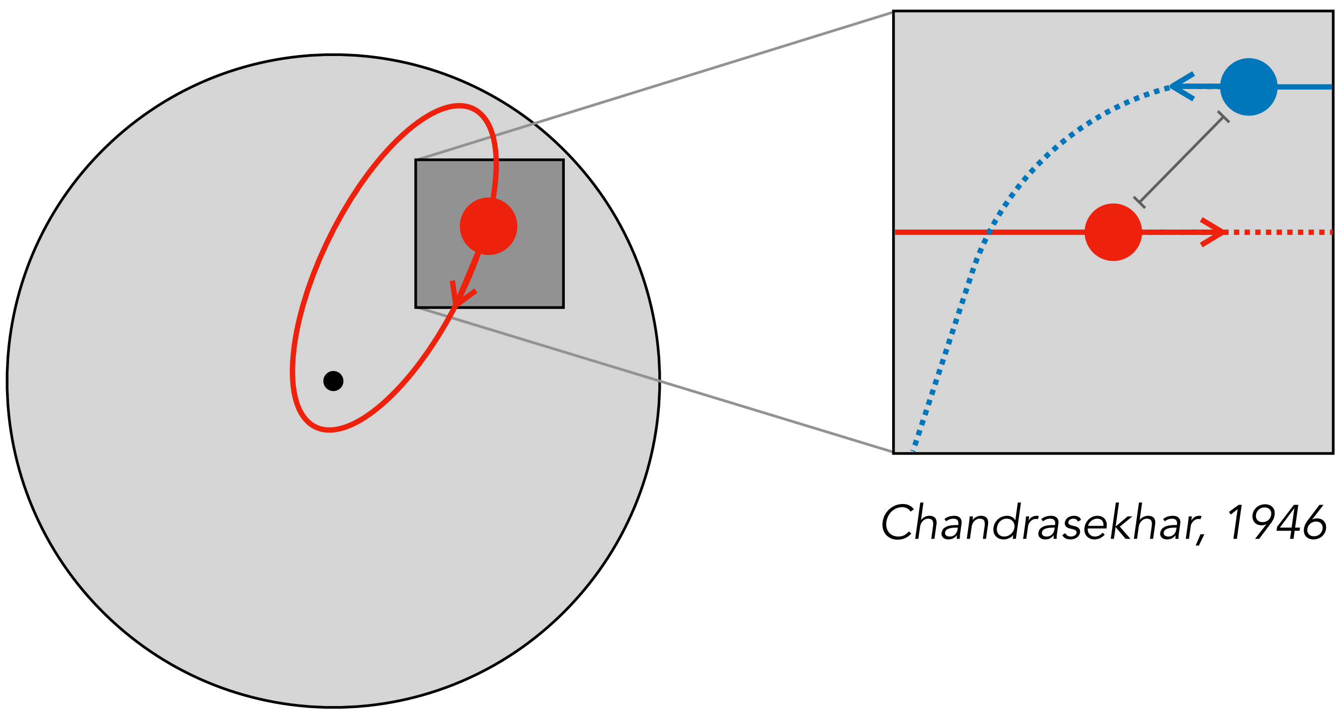
Random walk on the sphere of $\hat{\mathbf{L}}$

Vector Resonant Relaxation



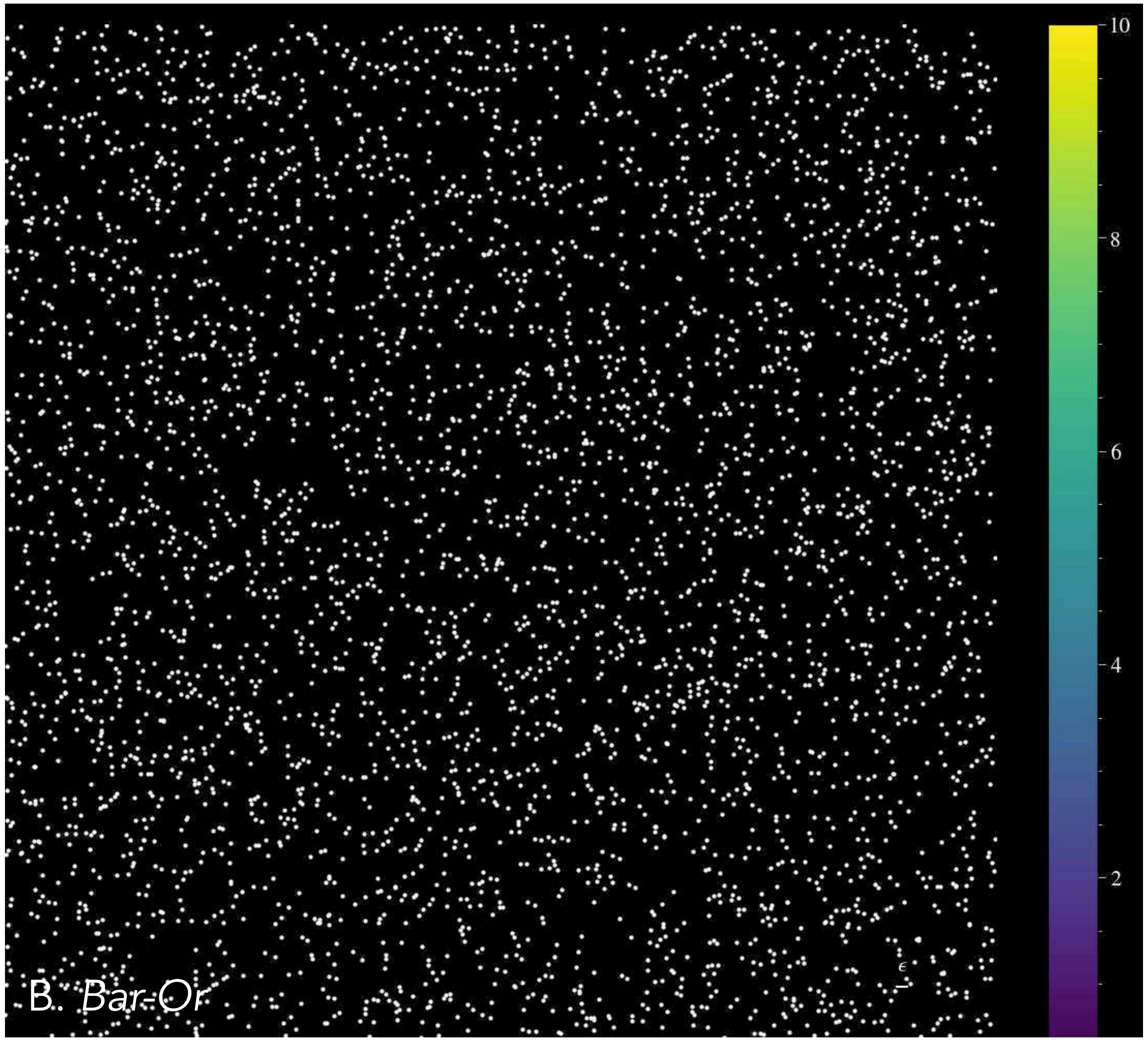
Non-Resonant Relaxation

Orbital distortions sourced by instantaneous **kicks and deflections**



See Hamilton's talk

Chandrasekhar, 1946



+ **Local**, **uncorrelated** and **non-resonant** encounters, i.e. slowest dynamics

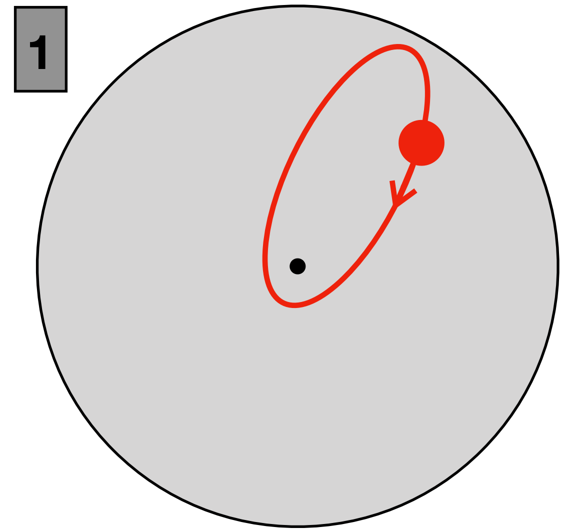
+ Immune to orbit-average and **adiabatic invariance**: $\dot{a} = \eta(t)$

Timescales are highly hierarchical

1. Dynamical time

Fast orbital motion induced by the BH

$$\frac{\mathrm{d}M}{\mathrm{d}t} = \Omega_{\mathrm{Kep}}$$



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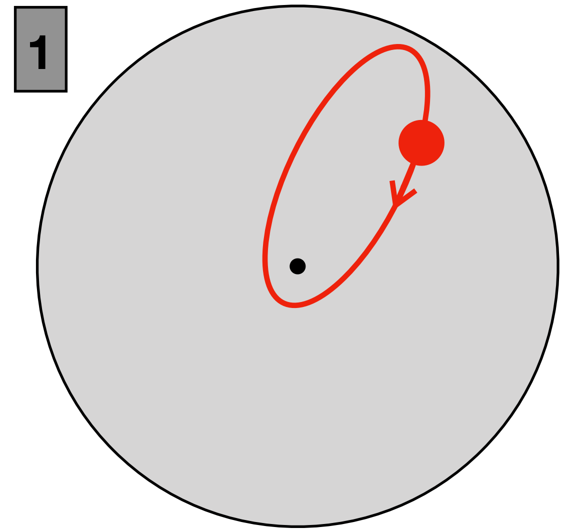
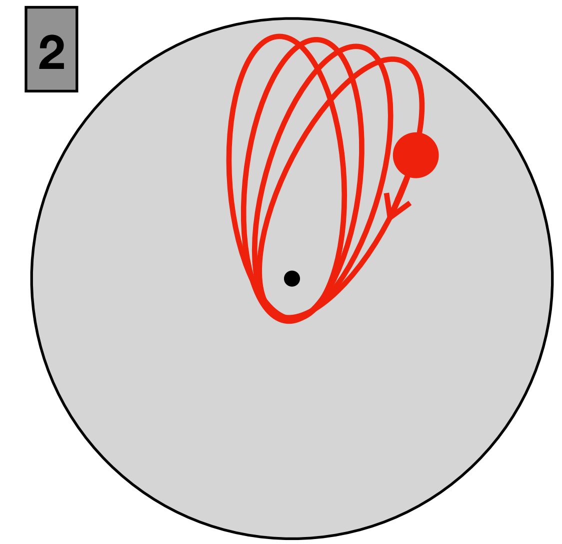
2. Precession time
In-plane precession (mass + relativity)

$$\frac{\mathrm{d}\omega}{\mathrm{d}t} = \Omega_{\mathrm{prec}}$$





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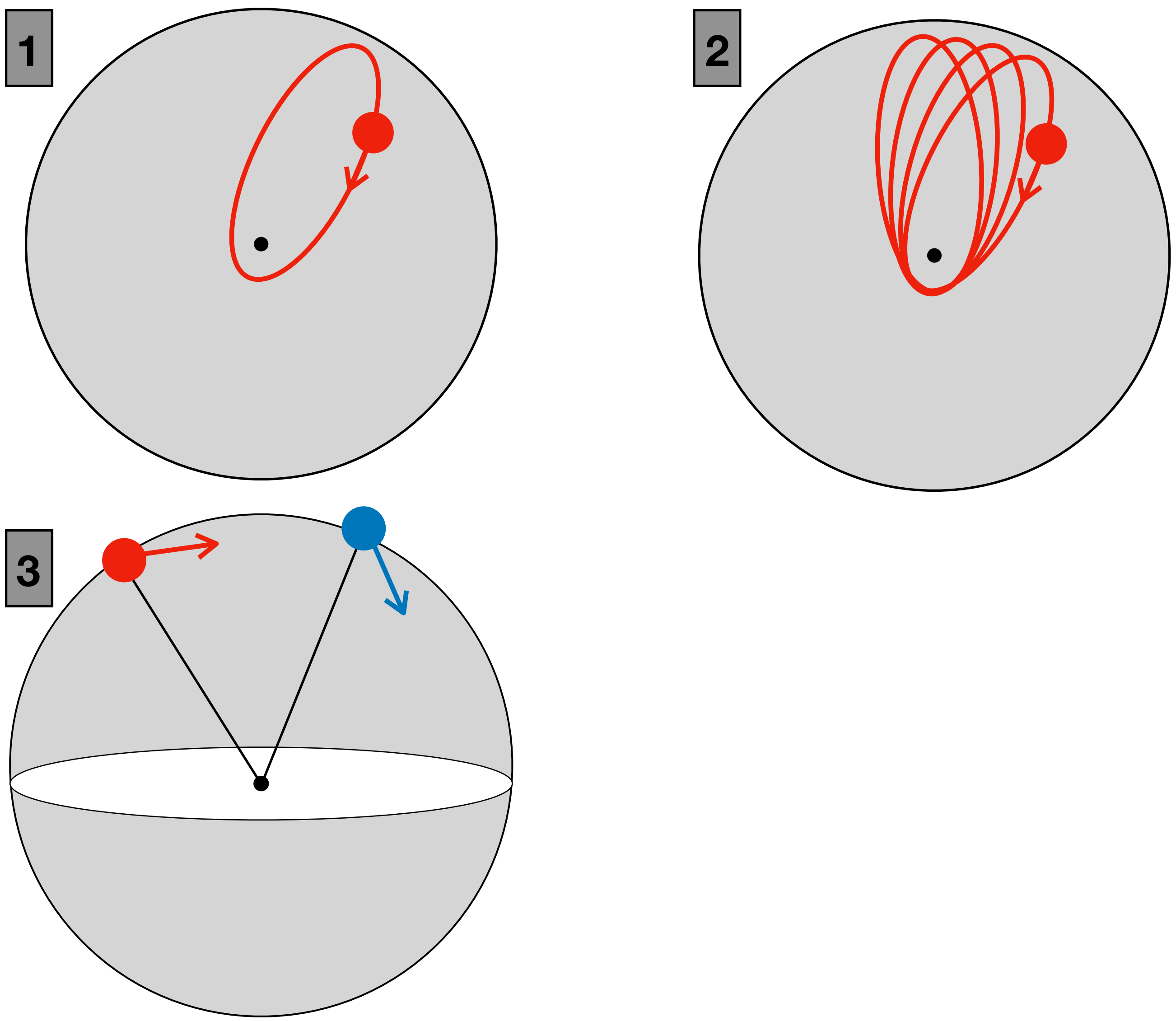
3. Vector Resonant Relaxation

Non-spherical torque coupling

$$\frac{\mathrm{d}\hat{\mathbf{L}}}{\mathrm{d}t} = \eta(\hat{\mathbf{L}}, t)$$



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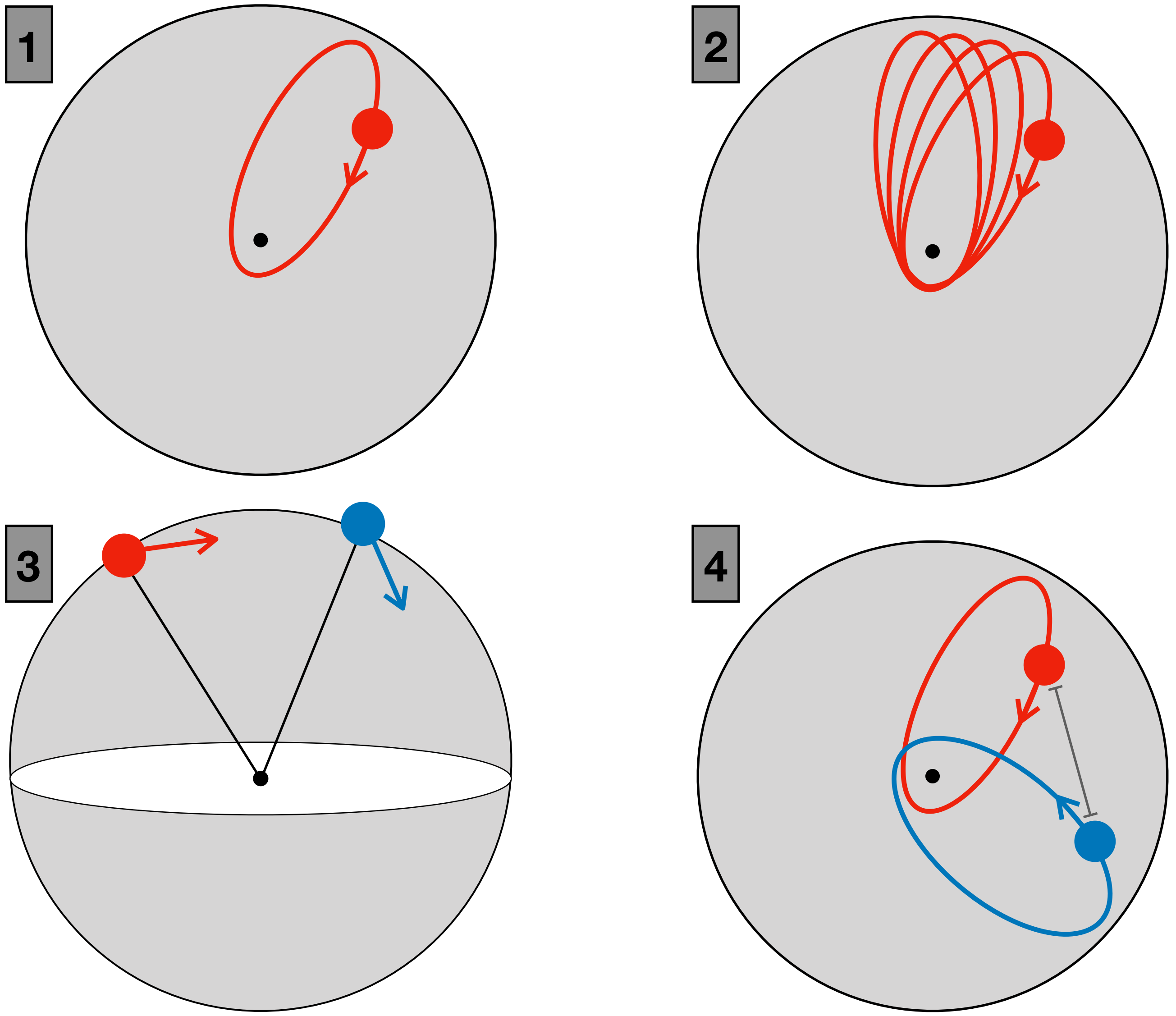
$$\frac{\mathrm{d}\hat{\mathbf{L}}}{\mathrm{d}t} = \eta(\hat{\mathbf{L}}, t)$$

4. Scalar Resonant Relaxation
Resonant coupling on precessions

$$\frac{\mathrm{d}\,|\mathbf{L}|}{\mathrm{d}t} = \eta(|\mathbf{L}|, t)$$



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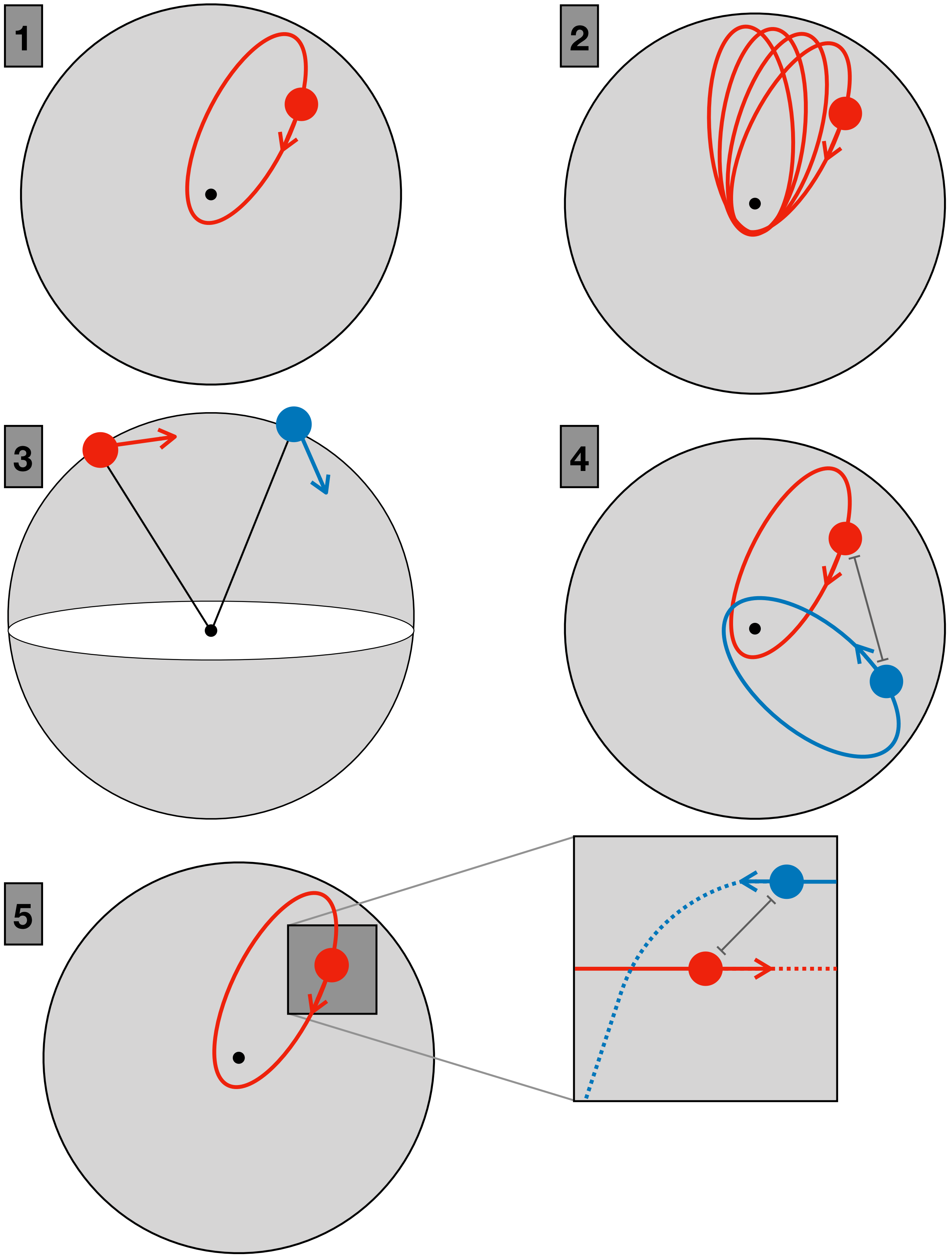
5. Non-Resonant Relaxation
Local two-body encounters

$$\frac{\mathrm{d}a}{\mathrm{d}t} = \eta(a, t)$$



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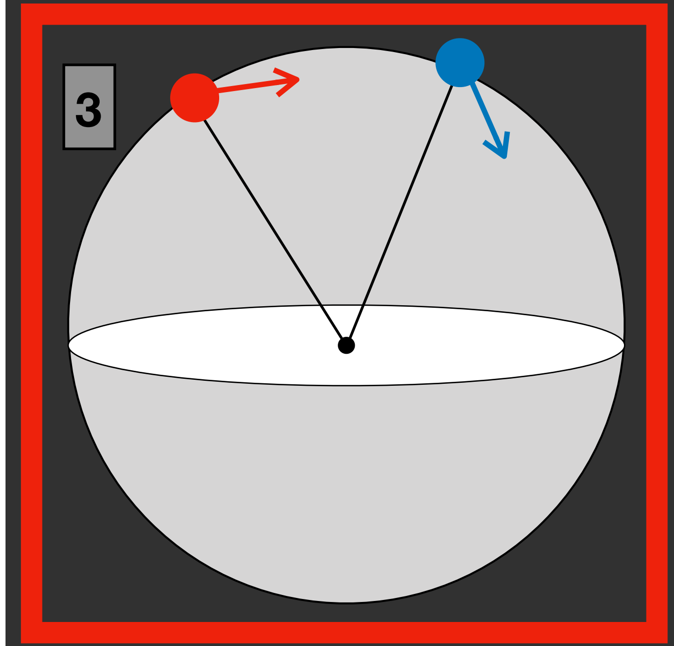
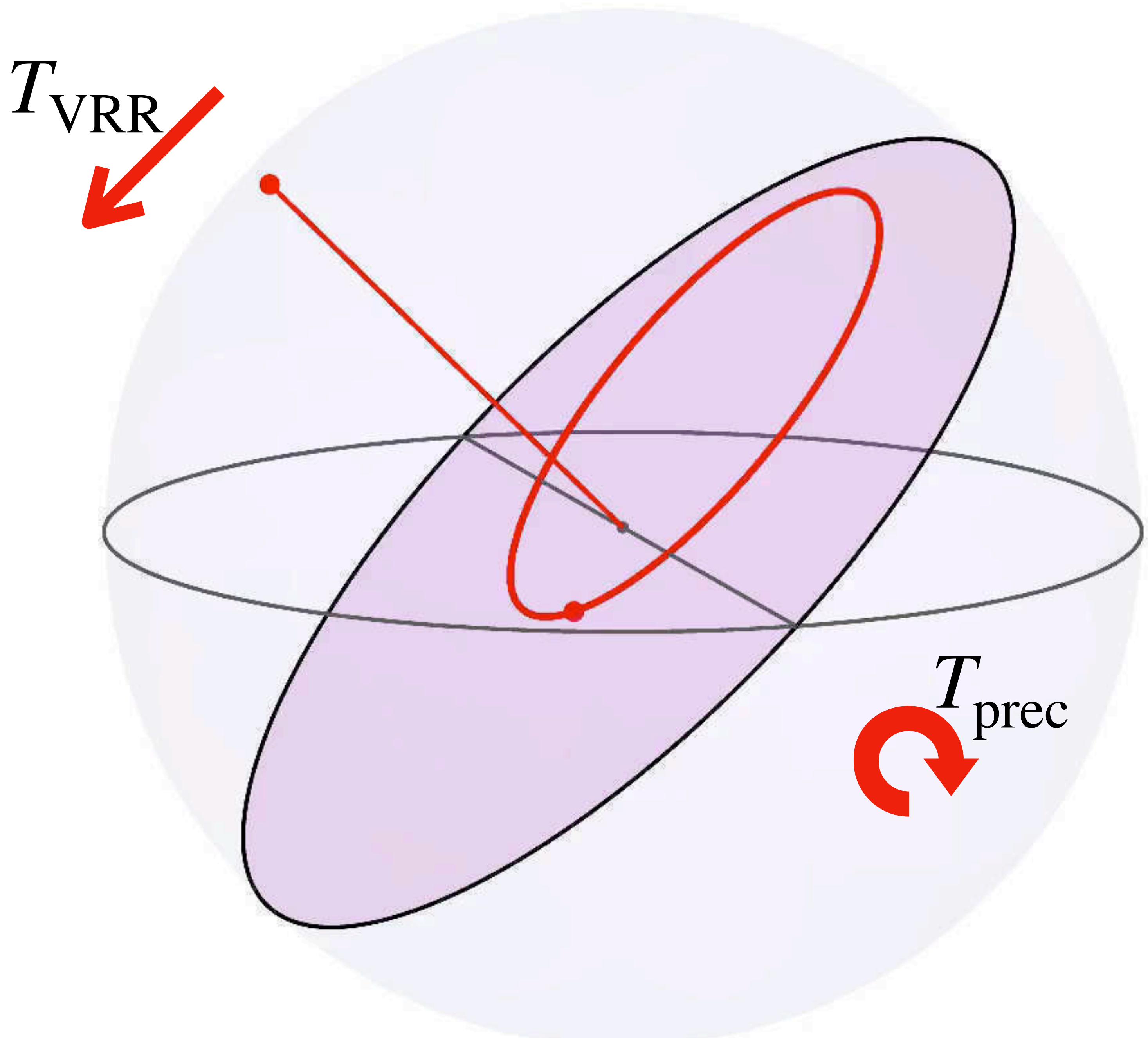


Vector Resonant Relaxation

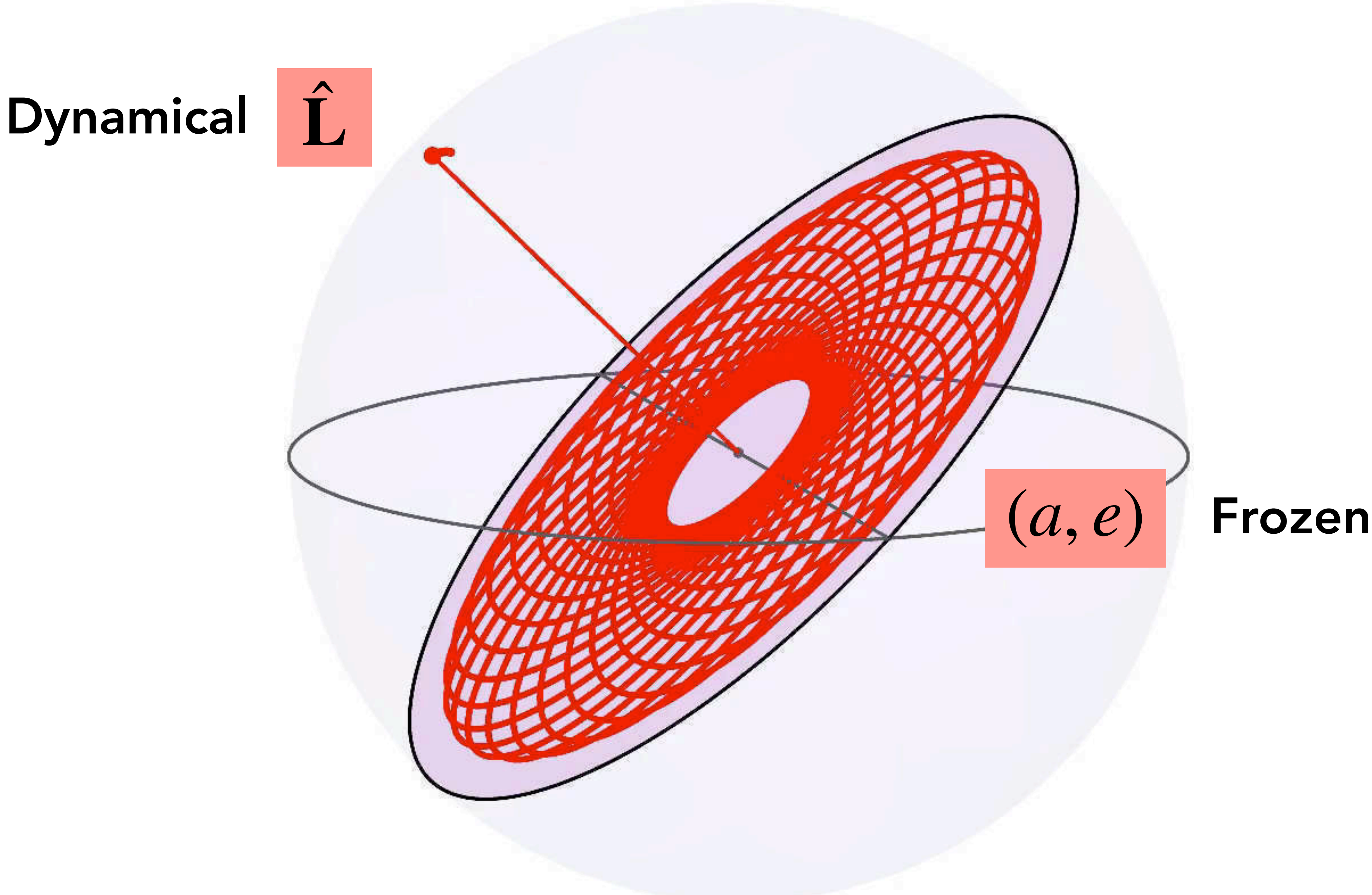
The dynamics of **Keplerian wires**



Since $T_{\mathrm{prec}} \ll T_{\mathrm{VRR}}$, we can perform a **second orbit-average**

Vector Resonant Relaxation

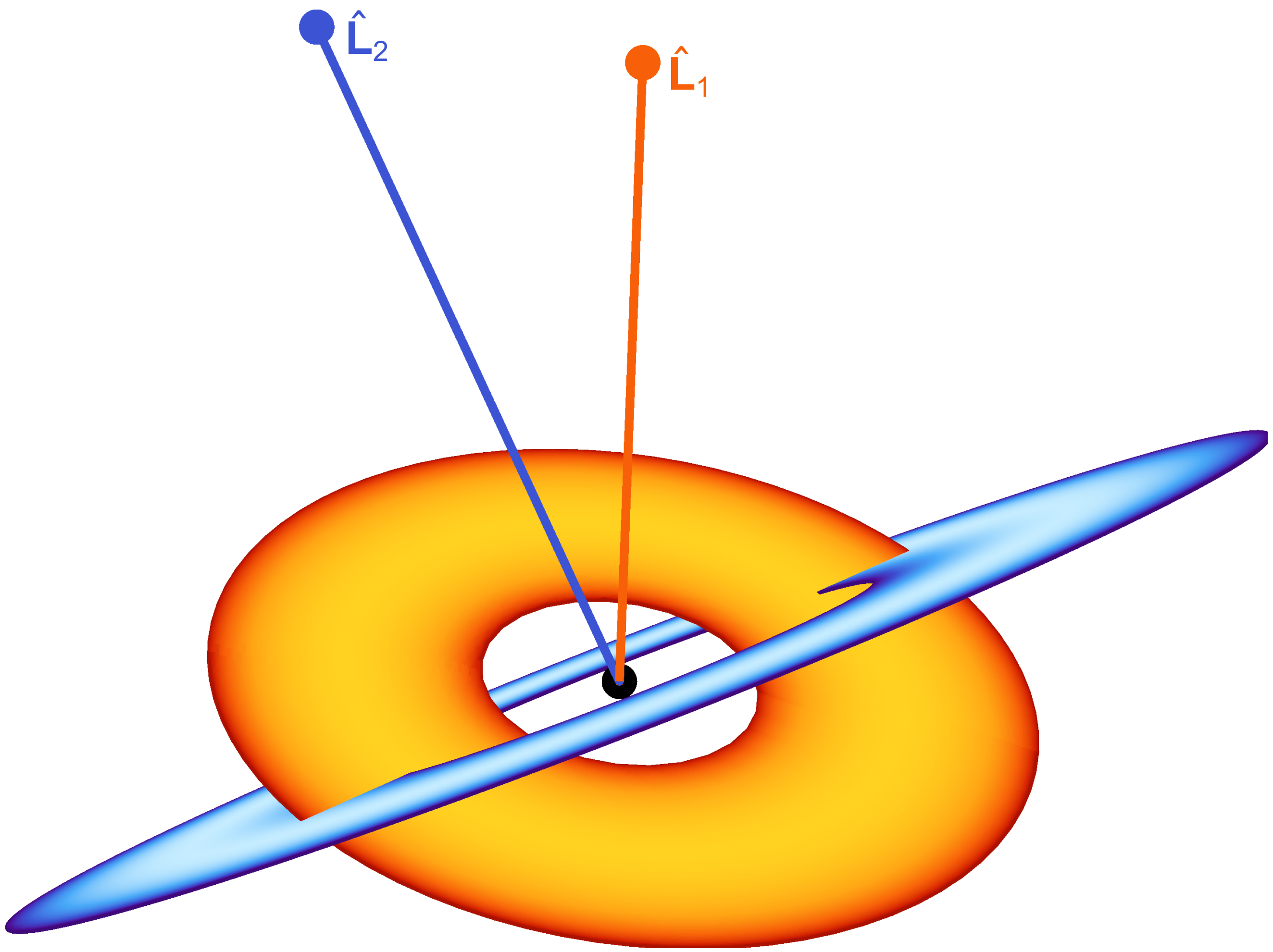
Orbit average: Wire $\Rightarrow$ Annuli



What is the dynamics of a set of **long-range coupled annuli**?

Vector Resonant Relaxation

Random walk of the stars' orientations



Pairwise coupling between two annuli

+ **Long-range** Hamiltonian system

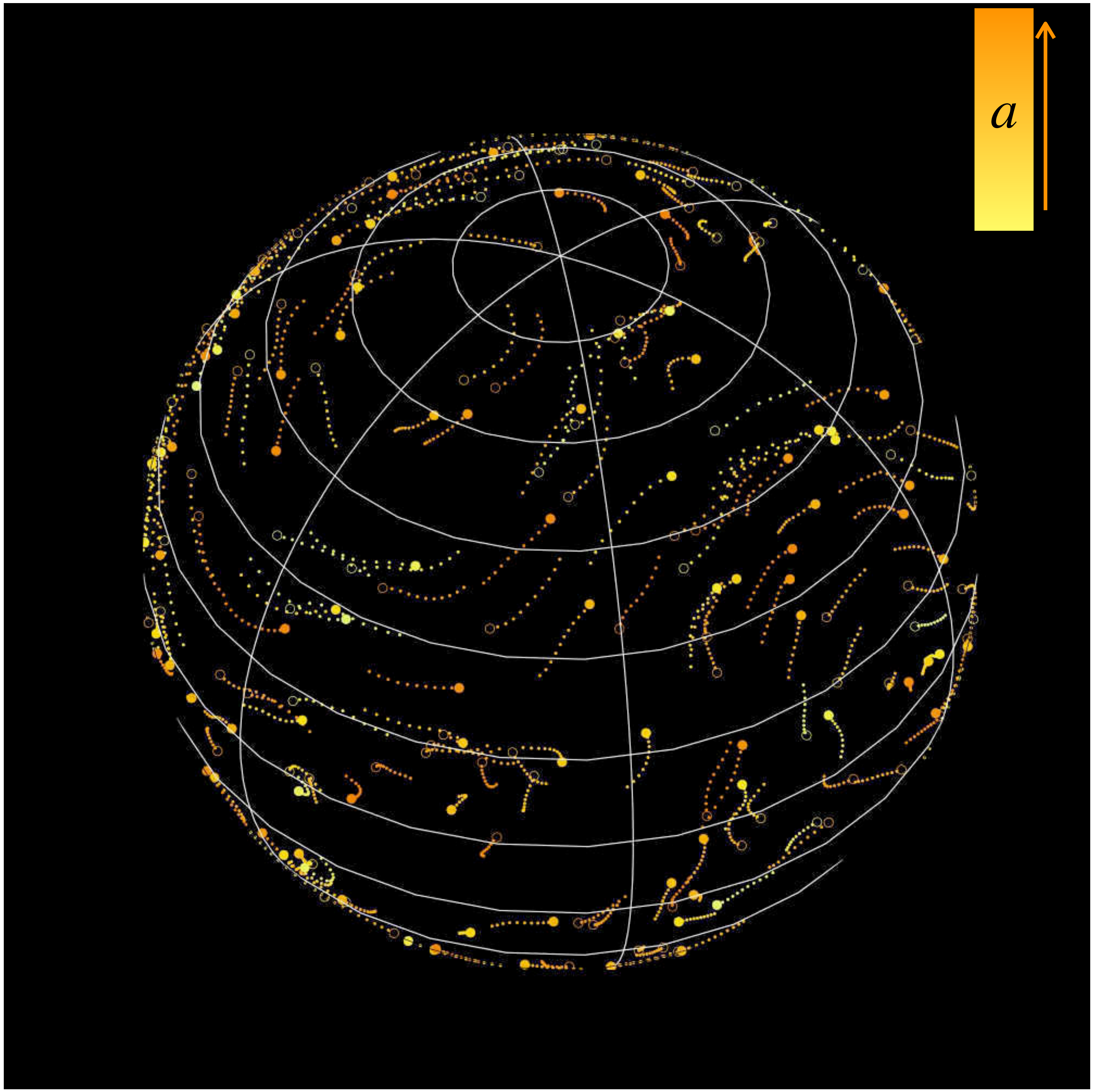
$$H = \sum_{i<j} A(a_i, e_i, a_j, e_j)\, U(\hat{\mathbf{L}}_i \cdot \hat{\mathbf{L}}_j)$$

+ Dynamical variables - **orientations**: $\hat{\mathbf{L}}$

+ Some properties

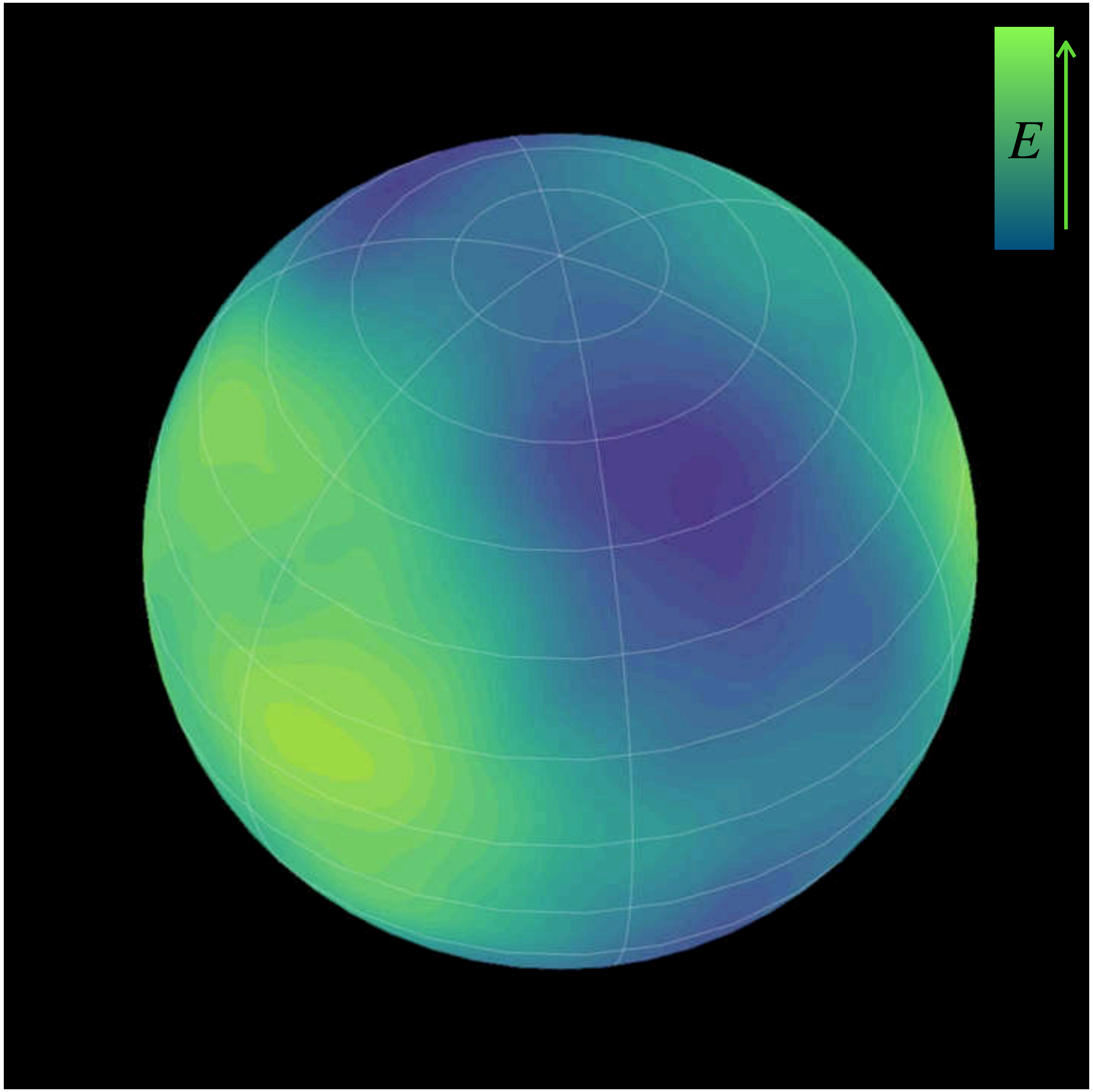
  - No **kinetic energy**
  - Vanishing **mean field** $\langle H \rangle = 0$
  - Additional "labels" (a, e)
  - **Rotational invariance** $\hat{\mathbf{L}}_i \cdot \hat{\mathbf{L}}_j$

Vector Resonant Relaxation



+ Motion coherent on large scales
    - **Long-range interacting system**
+ Motion smooth on short times
    - **Time-correlated noise**
+ Particles have "preferred friends"
    - **Parametric coupling** (a, e)
+ System in statistical equilibrium
    - **Time stationarity** $(t - t')$
    - **Rotation invariance** $(\hat{\mathbf{L}} \cdot \hat{\mathbf{L}}')$

Vector Resonant Relaxation



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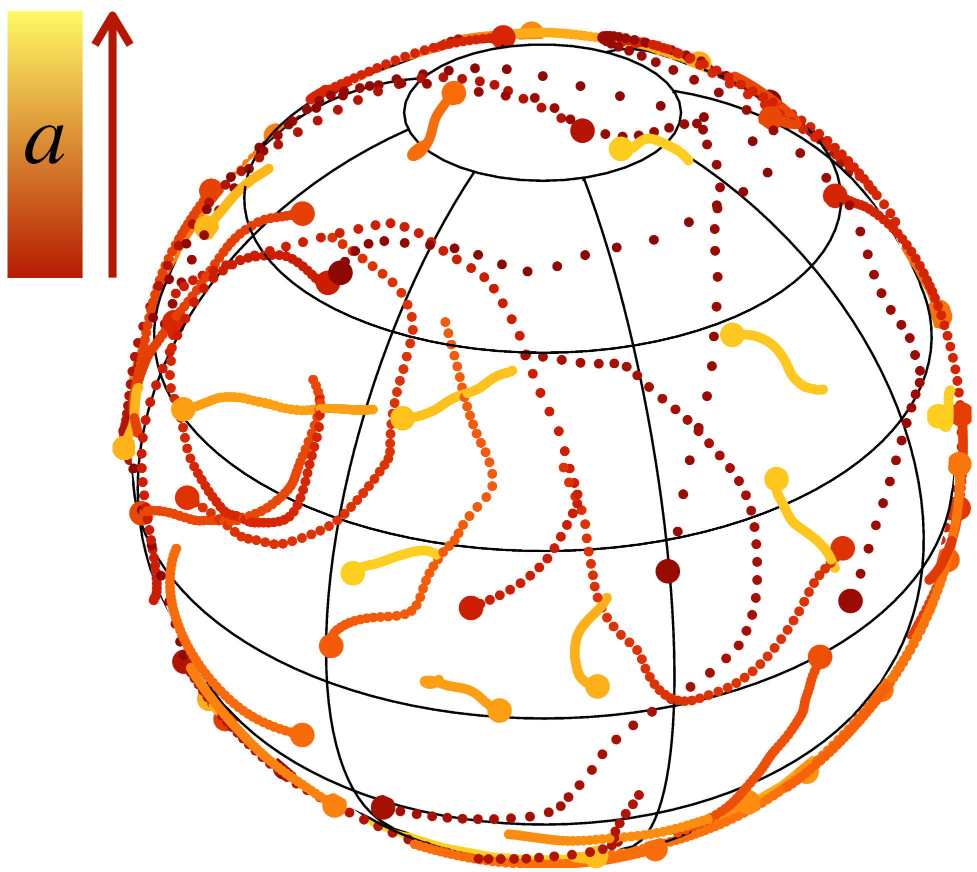
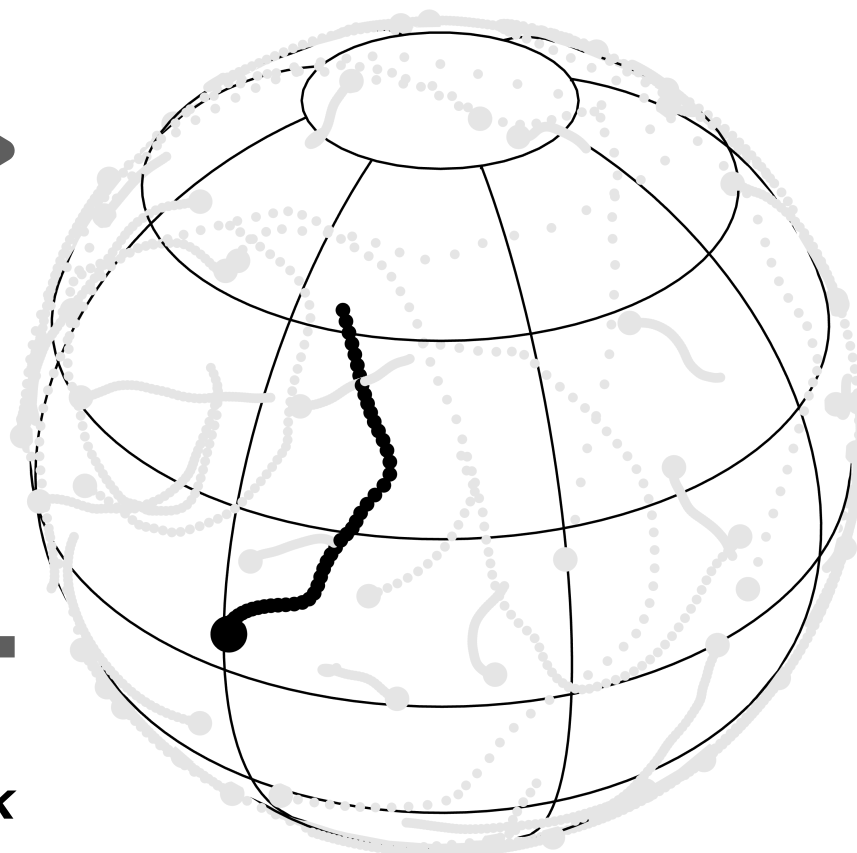
Self-consistency requirement

Bath of particles



Imposes a noisy (correlated) ***potential***

Test particle

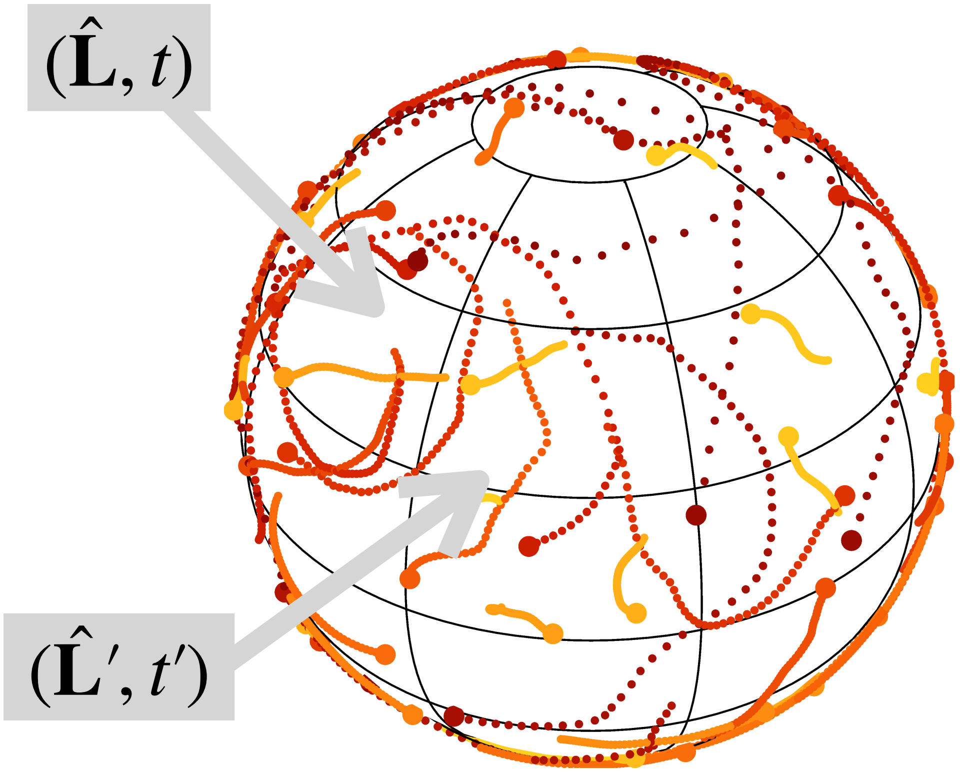


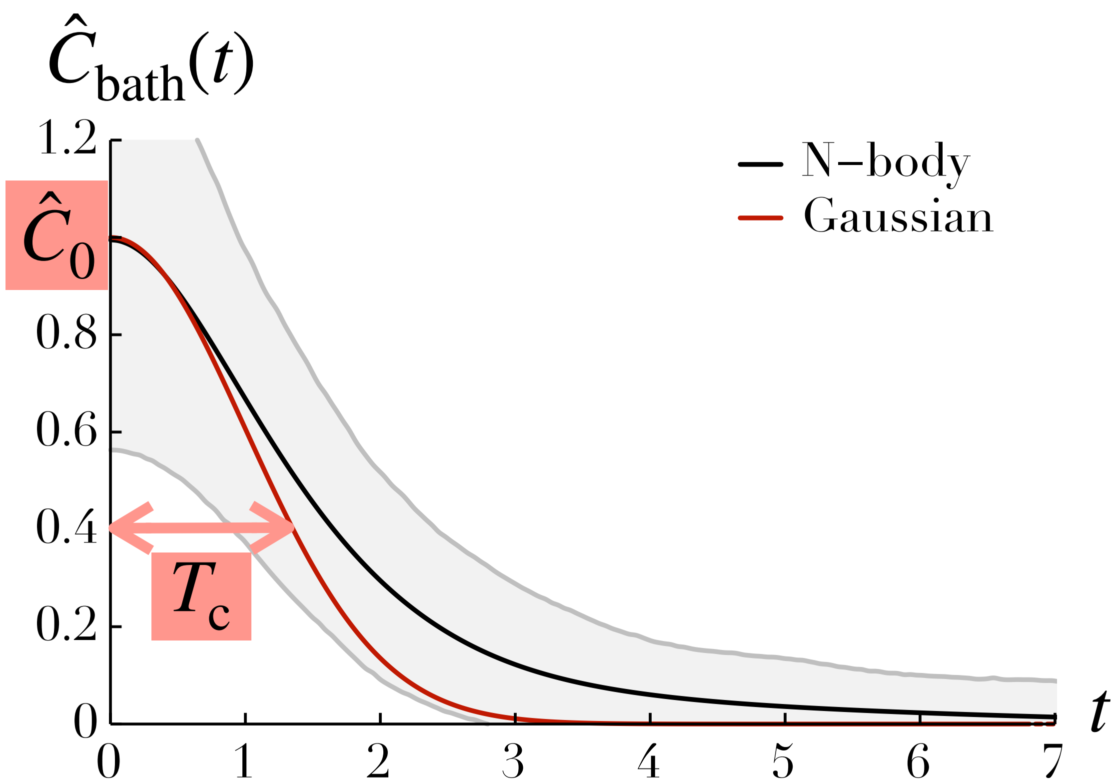
Undergoes a (correlated) ***random walk***

$$\hat{C}_{\mathrm{bath}} = \left\langle \eta(\hat{\mathbf{L}}, t)\, \eta(\hat{\mathbf{L}}', t') \right\rangle$$

$$\hat{C}_{\mathrm{test}} = \left\langle \hat{\mathbf{L}}_{\mathrm{test}}(t) \cdot \hat{\mathbf{L}}_{\mathrm{test}}(0) \right\rangle$$

Characterising the bath noise $\hat{C}_{\mathrm{bath}} = \left\langle \eta(\hat{\mathbf{L}}, t)\, \eta(\hat{\mathbf{L}}', t') \right\rangle$





+ The **state of the bath** is fully characterised by

$$\varphi_{\mathrm{bath}}(\hat{\mathbf{L}}, t) = \frac{1}{N} \sum_{i=1}^{N} \delta_{\mathrm{D}}(\hat{\mathbf{L}} - \hat{\mathbf{L}}_i(t))$$

+ System's (quadratic) **evolution equation**

$$\frac{\partial \varphi_{\mathrm{bath}}(t)}{\partial t} = Q\, \varphi_{\mathrm{bath}}(t)\, \varphi_{\mathrm{bath}}(t)$$

+ Good news
  - At t=0, particles are **statistically decorrelated**
  - Very constraining **spherical symmetries**

+ **Initial time statistics**

$$\left\langle \hat{C}_{\mathrm{bath}}(t=0) \right\rangle \qquad \left\langle \left. \frac{\mathrm{d}^2 \hat{C}_{\mathrm{bath}}}{\mathrm{d}t^2} \right|_{t=0} \right\rangle$$

Initial amplitude — *Coherence time*

+ (Natural) **Gaussian** Ansatz

$$\hat{C}_{\mathrm{bath}}(t) = \hat{C}_0\, \mathrm{e}^{-(t/T_{\mathrm{c}})^2}$$

Vlasov?

Discrete PDF

$$\varphi_{\mathrm{bath}}(\hat{\mathbf{L}}, t) = \frac{1}{N} \sum_{i=1}^{N} \delta_{\mathrm{D}}(\hat{\mathbf{L}} - \hat{\mathbf{L}}_i(t))$$

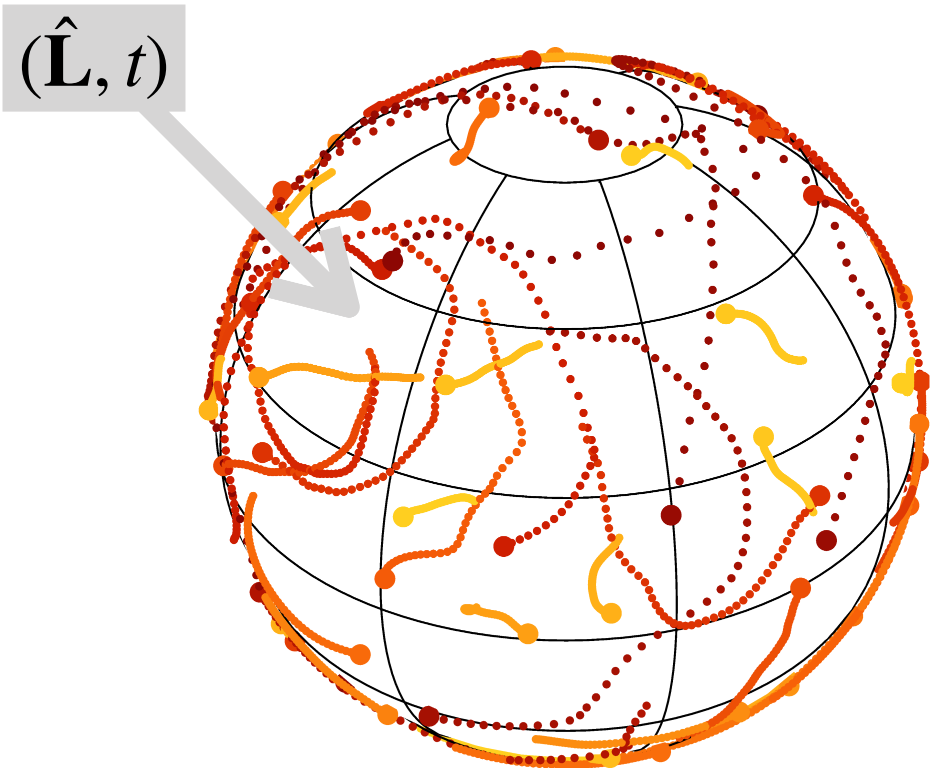
Continuity equation (i.e. **Klimontovich** equation)

$$\frac{\partial \varphi_{\mathrm{bath}}}{\partial t} + \left[\varphi_{\mathrm{bath}} \, , H[\varphi_{\mathrm{bath}}] \right] = 0$$

with the **self-consistency**

$$H[\varphi]\big(\hat{\mathbf{L}}\big) = \int \mathrm{d}\hat{\mathbf{L}}' \, \psi(\hat{\mathbf{L}}, \hat{\mathbf{L}}') \, \varphi(\hat{\mathbf{L}}')$$

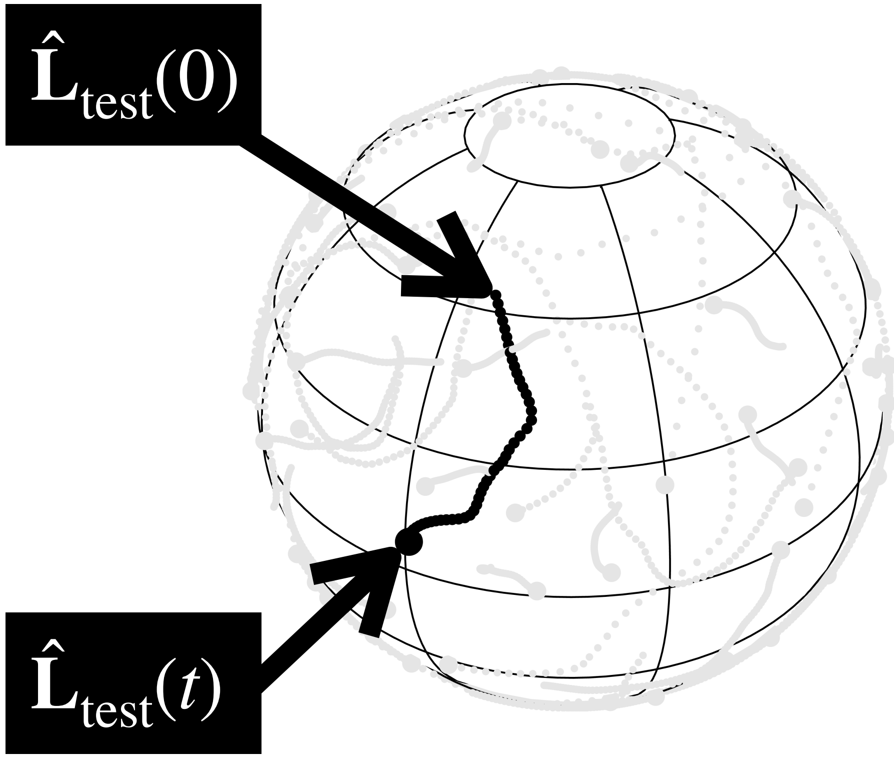
Bath of particles

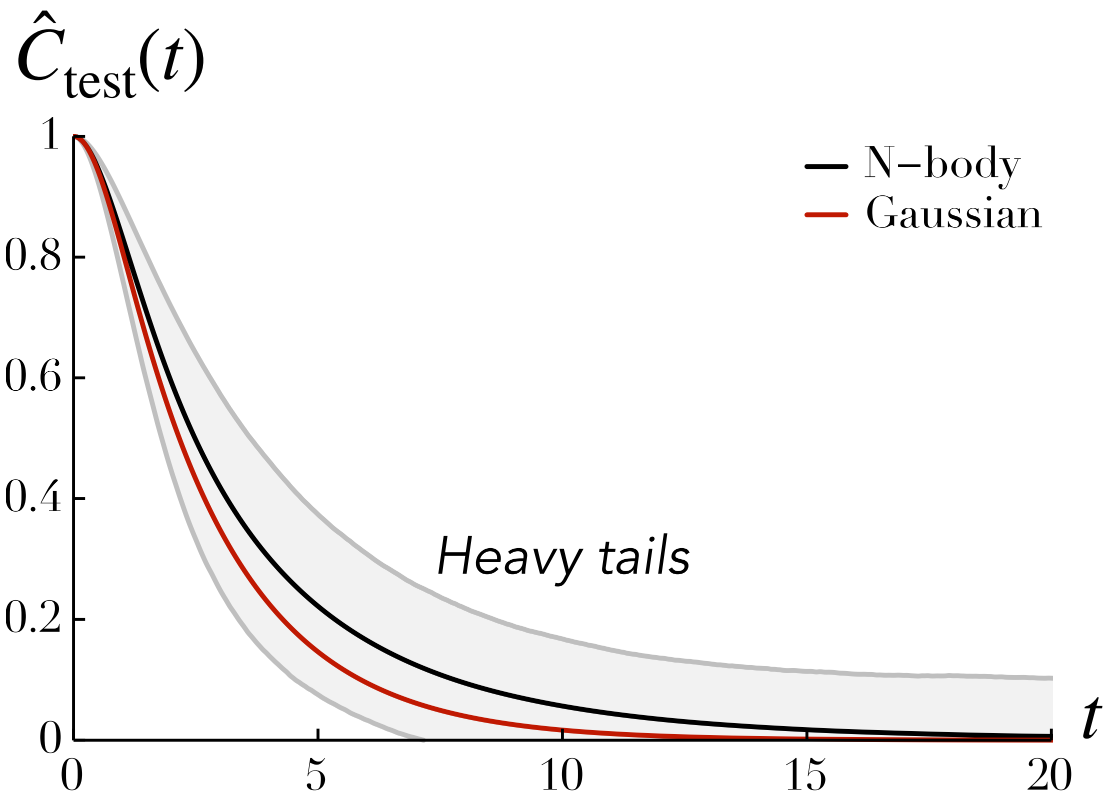


Only difference with (collisionless) Vlasov is the **statistics**

$$\Big\langle \varphi_{\mathrm{bath}}\big(\hat{\mathbf{L}}, t\big) \, \varphi_{\mathrm{bath}}\big(\hat{\mathbf{L}}', t'\big) \Big\rangle \; \neq \; \Big\langle \varphi_0\big(\hat{\mathbf{L}}, t\big) \, \varphi_0\big(\hat{\mathbf{L}}', t'\big) \Big\rangle$$

Characterising the random walk $\hat{C}_{\mathrm{test}} = \left\langle \hat{\mathbf{L}}_{\mathrm{test}}(t) \cdot \hat{\mathbf{L}}_{\mathrm{test}}(0) \right\rangle$





+ Location of the **test particle** characterised by

$$\varphi_{\mathrm{test}}(\hat{\mathbf{L}}, t) = \delta_{\mathrm{D}}(\hat{\mathbf{L}} - \hat{\mathbf{L}}_{\mathrm{test}}(t))$$

+ (Linear) **time-dependent** evolution equation

$$\frac{\partial \varphi_{\mathrm{test}}(t)}{\partial t} = \eta_{\mathrm{bath}}(t)\, \varphi_{\mathrm{test}}(t)$$

+ Good news
  - Noise is treated as **external**
  - Very constraining **spherical symmetry**

+ Motion solved using **Magnus series**

$$\varphi_{\mathrm{test}}(t) = \mathrm{e}^{\Omega(t)}\, \varphi_{\mathrm{test}}(0) \quad \text{with} \quad \Omega(t) = \int_0^t \mathrm{d}t'\, \eta_{\mathrm{bath}}(t')$$

+ Explicit expression of the **time correlation**

$$\hat{C}_{\mathrm{test}}(t) = \exp\left[- \int_0^t \mathrm{d}t_1 \int_0^t \mathrm{d}t_2\, \hat{C}_{\mathrm{bath}}(t_1 - t_2) \right]$$

Vlasov?

Test DF

$$\varphi_{\mathrm{test}}(\hat{\mathbf{L}}, t) = \delta_{\mathrm{D}}\big(\hat{\mathbf{L}} - \hat{\mathbf{L}}_{\mathrm{test}}(t)\big)$$

From **Klimontovich** to **Fokker-Planck**

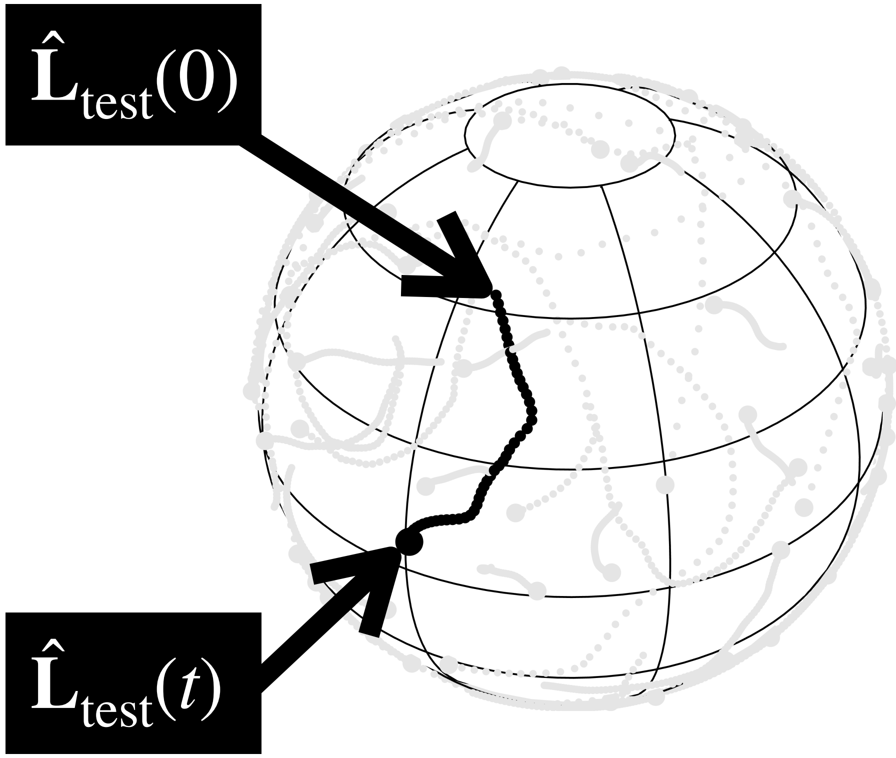
$$\frac{\partial \varphi_{\mathrm{bath}}}{\partial t} + \Big[\varphi_{\mathrm{bath}}\,,\, H[\varphi_{\mathrm{bath}}]\Big] = 0$$

$$\downarrow$$

$$\frac{\partial \varphi_{\mathrm{test}}}{\partial t} + \Big[\varphi_{\mathrm{test}}\,,\, H[\varphi_{\mathrm{bath}}]\Big] = 0$$

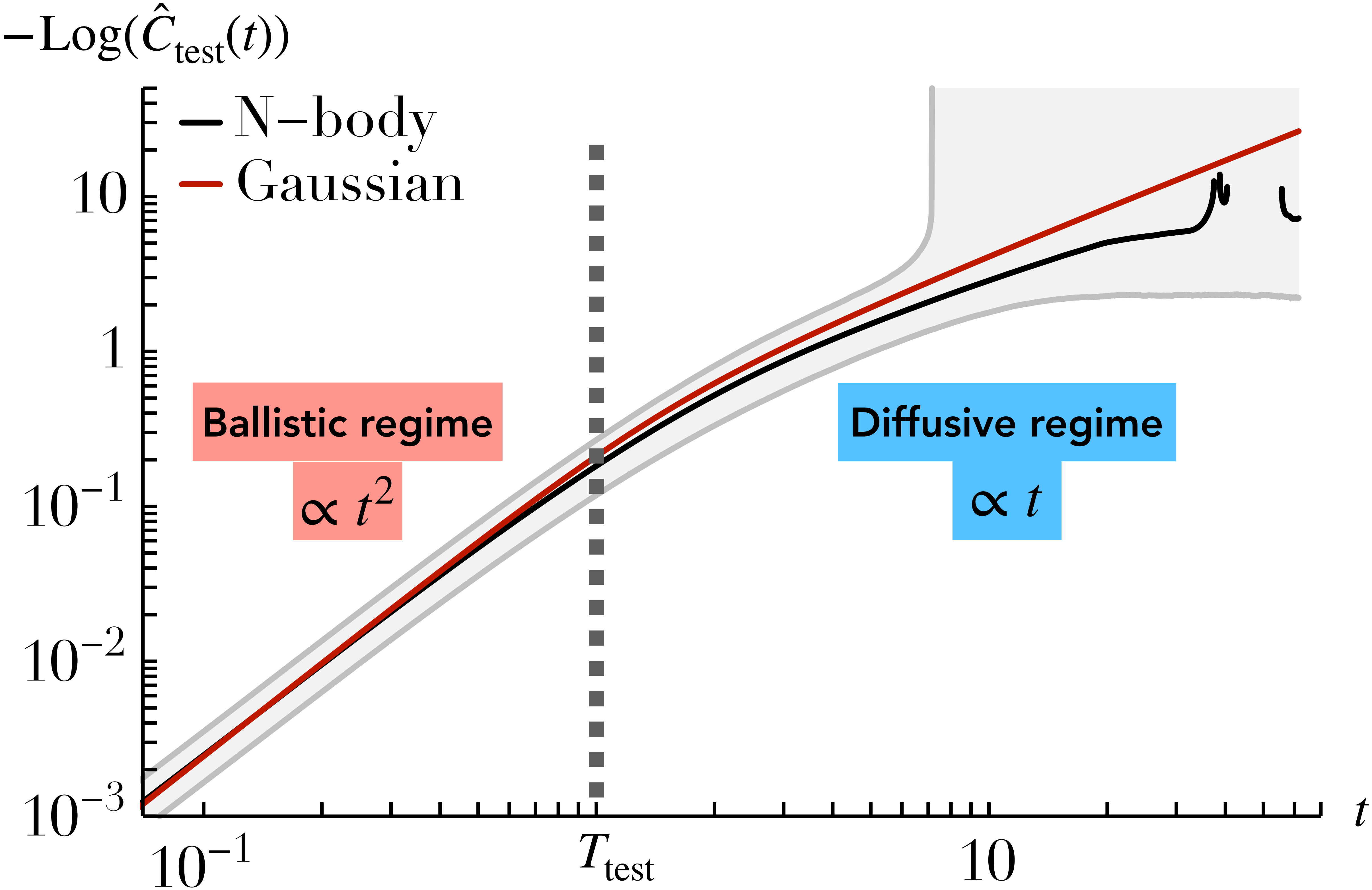
Externally-imposed correlated noise

Test particle



Characterising the random walk

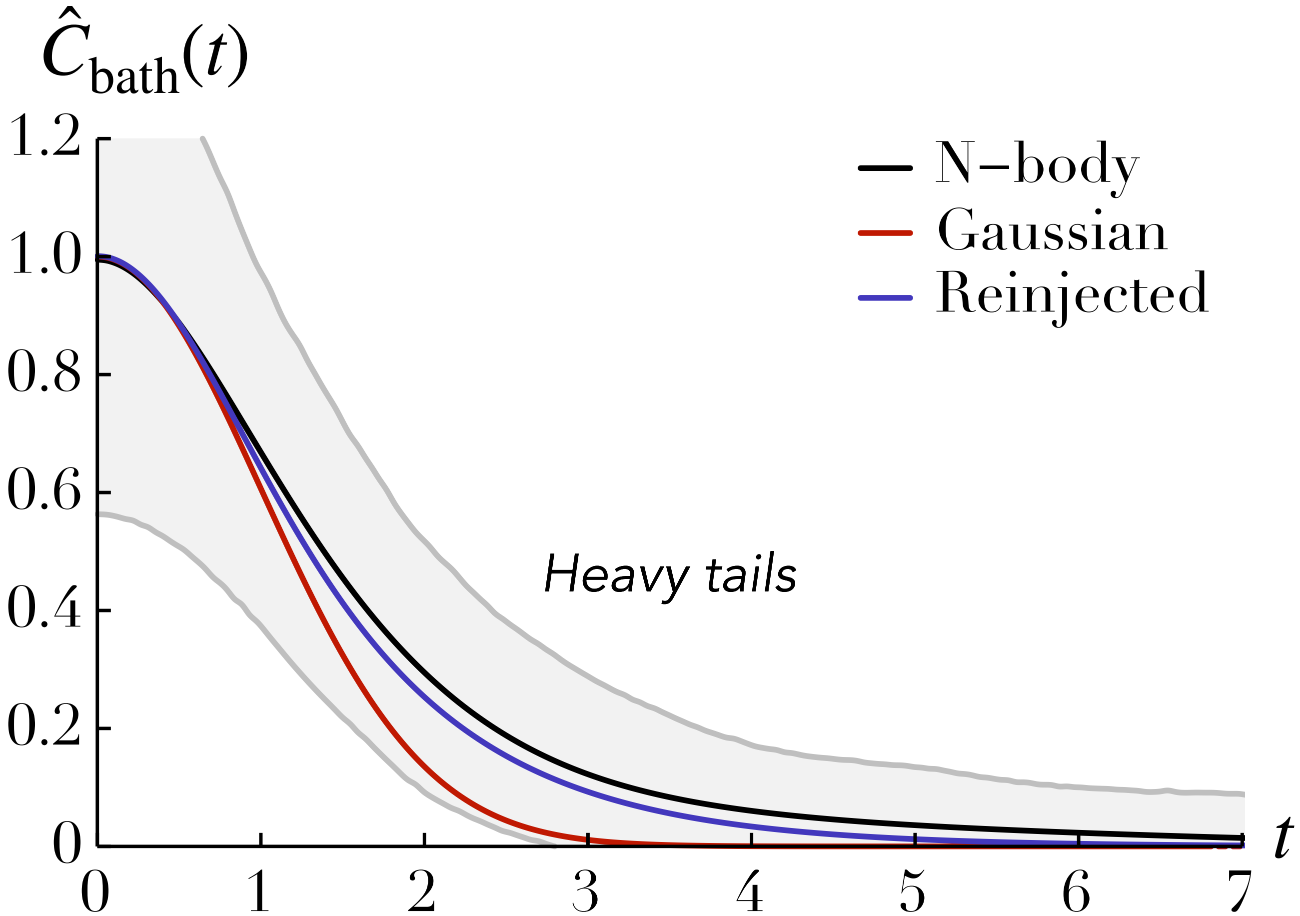
$$\hat{C}_{\rm test} = \left\langle \hat{\mathbf{L}}_{\rm test}(t) \cdot \hat{\mathbf{L}}_{\rm test}(0) \right\rangle$$



Can we do better?

Using self-consistency

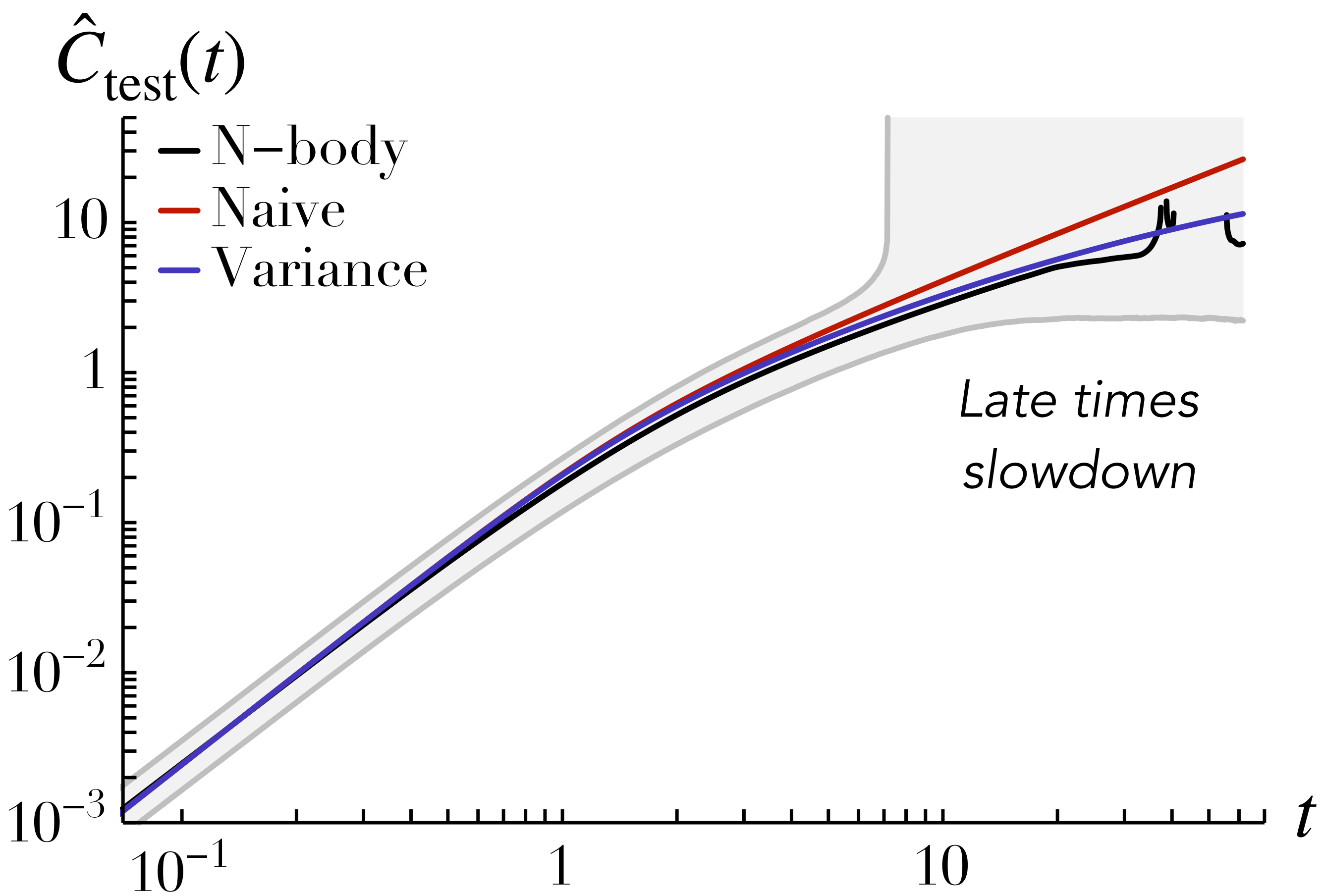
$$\begin{cases} \dfrac{\mathrm{d}\hat{C}_{\mathrm{test}}}{\mathrm{d}t} = \dfrac{\mathrm{d}\hat{C}_{\mathrm{test}}}{\mathrm{d}t}\big[\hat{C}_{\mathrm{bath}}\big] \\ \hat{C}_{\mathrm{bath}} = \Big\langle \hat{C}_{\mathrm{test}} \Big\rangle_{\text{All particles}} \end{cases}$$



Improving late times

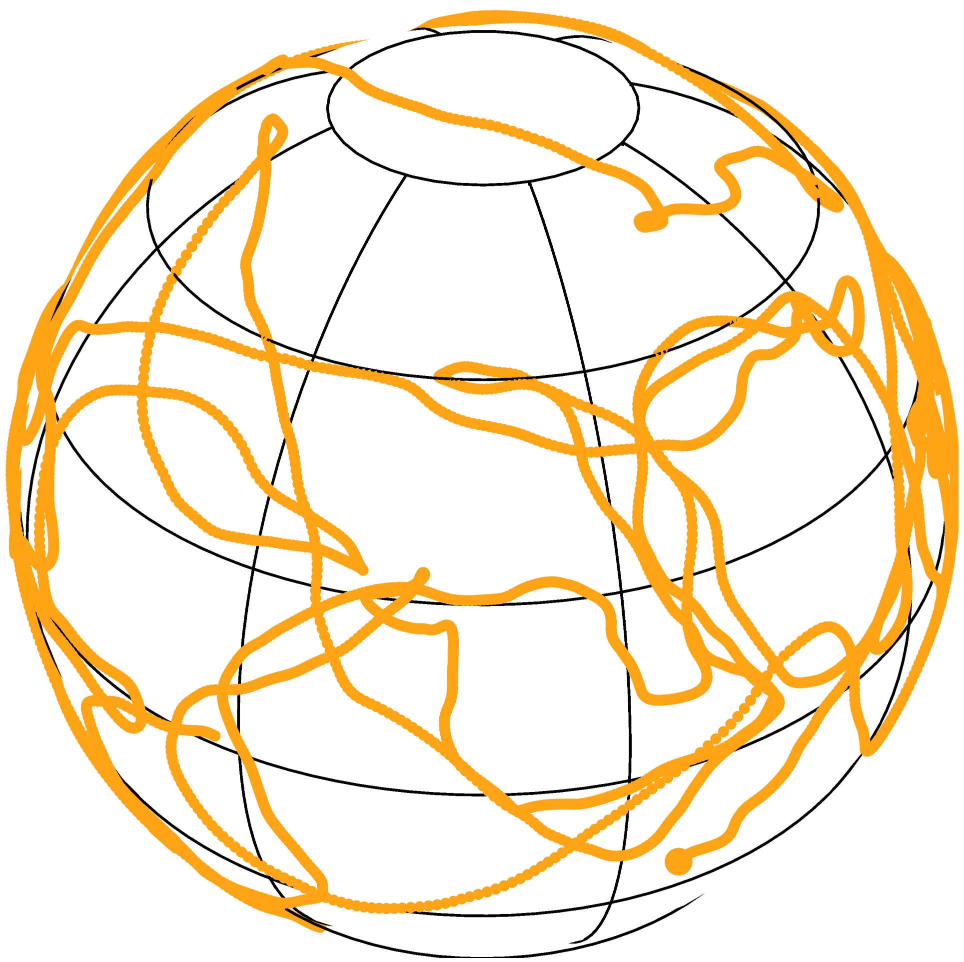
Stationary in time under the constraint
$(N, E_{\mathrm{tot}}, \mathbf{L}_{\mathrm{tot}})$ *conserved*

$$\langle \cdot \rangle_{\mathrm{Time}} \neq \langle \cdot \rangle_{\mathrm{Ensemble}}$$



Mimicking the random walk

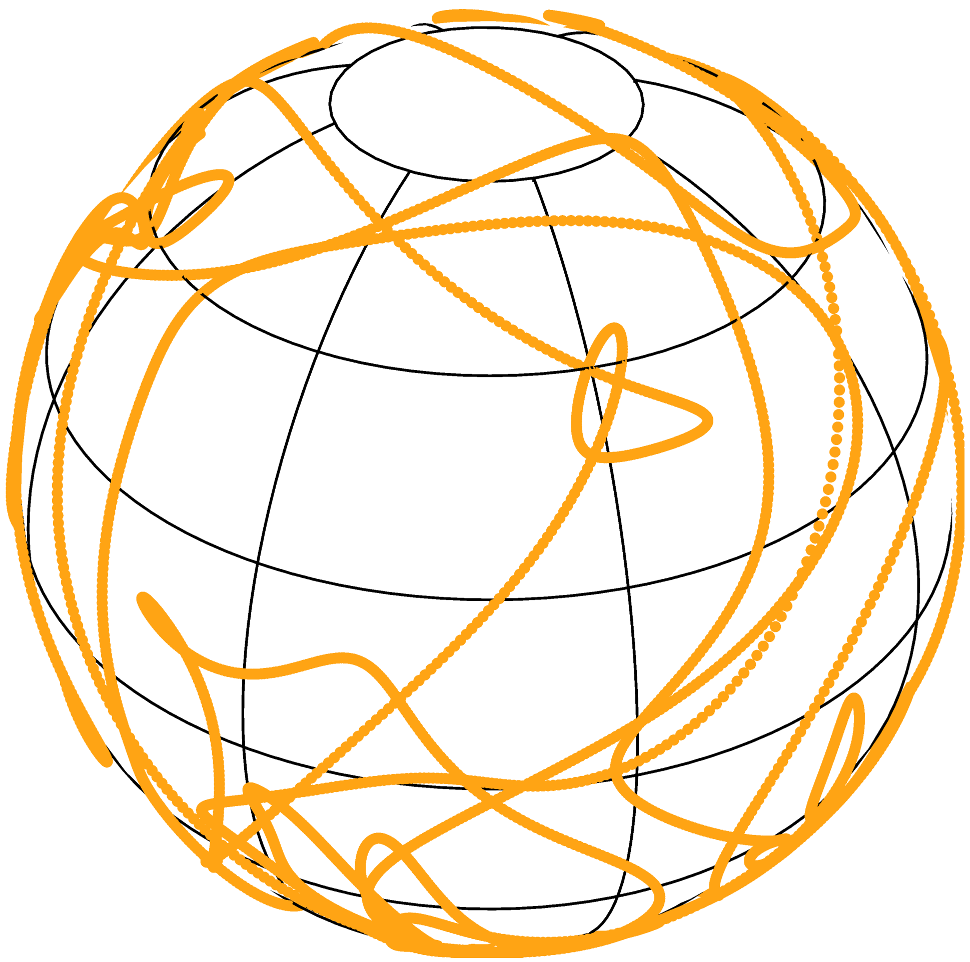
N-body simulations



$$H = \sum_{i<j}^{N} A_{ij}\, U(\hat{\mathbf{L}}_i \cdot \hat{\mathbf{L}}_j)$$

Full N-body problem of $\mathcal{O}(N^2)$ complexity

Effective model

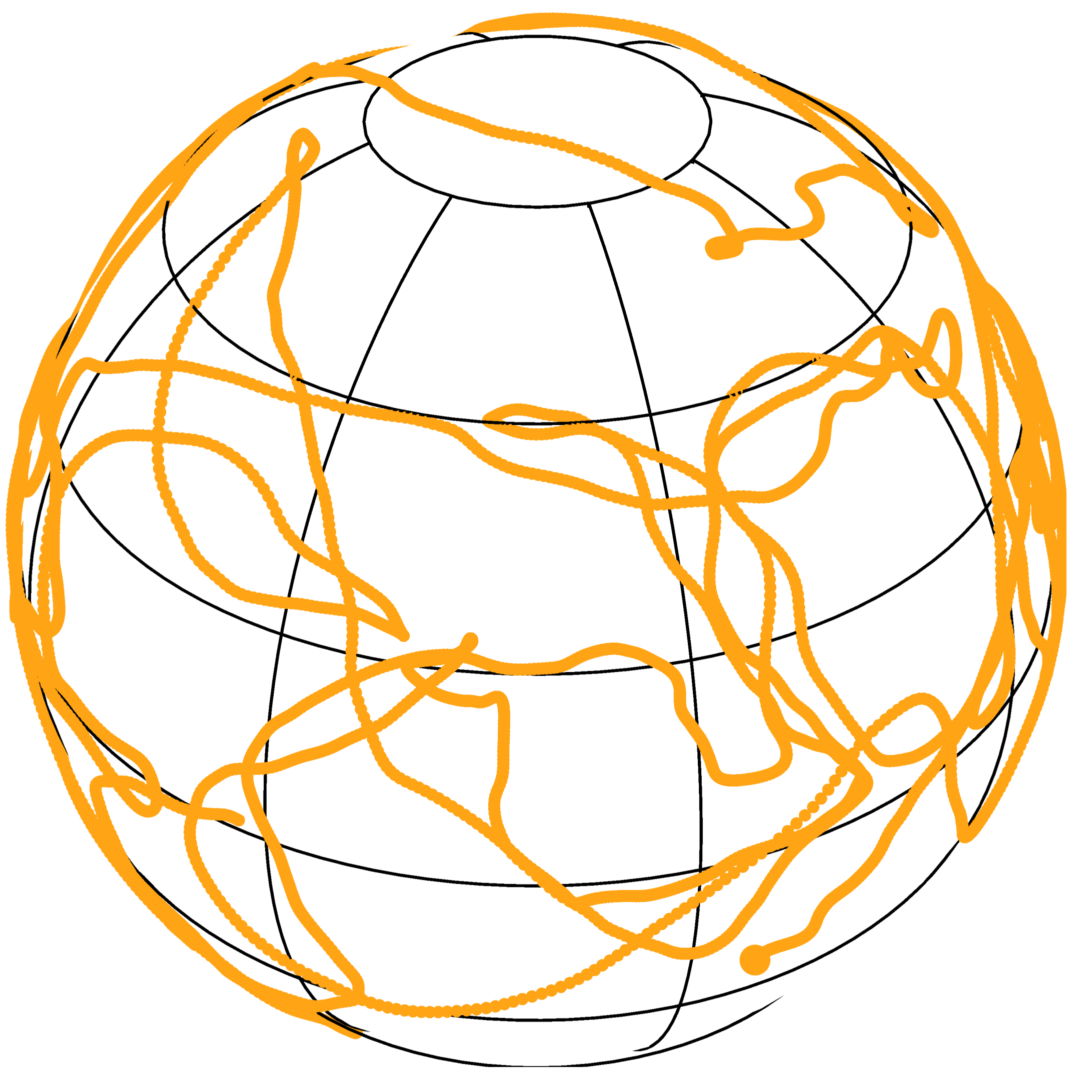


$$\frac{\mathrm{d}\hat{\mathbf{L}}_{\mathrm{test}}}{\mathrm{d}t} = \Gamma_{\mathrm{test}}\, \boldsymbol{\eta}(t) \times \hat{\mathbf{L}}_{\mathrm{test}}(t)$$

with
$$\begin{cases} \Gamma_{\mathrm{test}} & \textit{Amplitude} \\ \langle \boldsymbol{\eta}(t)\, \boldsymbol{\eta}(t') \rangle = \mathrm{e}^{-((t-t')/T_{\mathrm{test}})^2} & \textit{Coherence time} \end{cases}$$

Resonant Relaxation in Galactic Nuclei

Vector RR



Context

How to aliment a **supermassive black hole**?

Stellar diffusion in galactic centers

+ ***Origin and structure*** *of SgrA**
+ *Relaxation in* ***eccentricity***, ***orientation***

Sources of **gravitational waves**

+ *BHs-binary mergers*
+ *TDEs, EMRIs*

Novelties

+ **New kinetic equations** written and implemented
+ Confronted to **astrophysical observations**
+ Theory in a regime inaccessible to simulations

Next steps

Galactic centers

Stellar capture rates

Gravitational waves sources

Galactic discs

Galactic Archeology

Radial Migration/Thickening

Globular clusters

Effect of velocity anisotropy

Effect of rotation

Dark Matter halo

Cusp-Core transition

Environmental forcing