Vlasov equation from Kaluza-Klein model and from the guiding center description of motion

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Introduction

- Newton law of gravitation $F = G\frac{m_1 m_2}{r^2}$
- Coulomb law: $F = \pm\frac{1}{4\pi\varepsilon_0}\frac{e_1 e_2}{r^2}$

▷ In General Relativity bodies follow geodesics in space-time.

▷ The space-time is a manifold whose geometry depends on the distribution of particles and their energy by the Einstein's equation.

▷ There is not a direct interaction between particles.

▷ In the weak field approximation the effect on the space-time geometry acts as if a two body interaction applies.

- Even if the Coulomb law is so similar to the Newton gravitational law, it is believed that charges doesn't follow geodesics on space-time under electromagnetic forces.

Can charges follow geodesics, not on space-time, but on the phase-space manifold under electromagnetic and gravitational forces?

The guiding center description of motion

- The trajectory of a charge in e.m. field is seen to be helicoidal.
- Assume that exists a reference point in phase-space, called the guiding center, whose coordinates are X, U, t, from which the particle motion is seen to be closed and periodic (a loop in space as in the velocity space).
- If $\gamma \in S^1$ is the gyro-phase then the particle coordinates (x^α, u^α) in (a flat) phase space are expressed by
$$x^\alpha = X^\alpha + \rho^\alpha(\gamma), \quad u^\alpha = U^\alpha + \nu^\alpha(\gamma),$$
with $\alpha = 0, 1, 2, 3$.
- The guiding center coordinates are $(t, X, \gamma, \varepsilon, \mu)$. Explicitly: the time, t, the guiding center position, X, the gyro-phase, γ, the particle energy, ε, and the magnetic moment, μ.
- The guiding center transformation $\mathcal{T}$ is the map from (τ, x, u) to $(t, X, \gamma, \varepsilon, \mu)$.
- The single particle lagrangian, L, is rewritten as
$$L = -p_\alpha u^\alpha = -P_\alpha U^\alpha - (m/e)\mu\gamma', \qquad \text{with} \quad P_\alpha = U_\alpha + (e/m)A_\alpha,$$
with $\alpha = 0, 1, 2, 3$ and summation convention.
- The equation of motion has to be obtained adopting the guiding center coordinates.

Hamiltonian formulation

- Define the e.m. canonical field by $(e/m)E_c = -\partial_t P - \nabla P_0$ and $(e/m)B_c = \nabla \times P$, and $b = B_c/|B_c|$.
- The Lagrange form is

$$\sigma = -d\varepsilon \partial_\varepsilon P \wedge dX - d\mu \partial_\mu P \wedge dX + dX \cdot B_c \times dX + (m/e) d\mu \wedge d\gamma$$

- with Hamiltonian form $\mathcal{H} = d\varepsilon \partial_\varepsilon P_0 + d\mu \partial_\mu P_0 - dX \cdot E_c$.
- the Poisson form is $\mathbf{\Pi} = (e/m)\partial_\mu \wedge \partial_\gamma + \nabla \cdot (B_c/B_c^2) \times \nabla - \partial_\varepsilon \wedge b \cdot \nabla$
- equivalent to the Lie bracket

$$\{F, G\} = (e/m)(\partial_\mu F \partial_\gamma G - \partial_\mu F \partial_\gamma G) + B_c^{-2} \nabla F \cdot B_c \times \nabla G + (b \cdot \nabla F \partial_\varepsilon G - \partial_\varepsilon F b \cdot \nabla G)$$

- The obtained equations of motion are

$$\dot{X} = V_b b + \frac{E_c \times B_c}{B_c^2}, \qquad \dot{\gamma} = -(e/m)\partial_\mu P_0, \qquad \dot{\varepsilon} = V_b \frac{b \cdot E_c}{\partial_\varepsilon P_0}, \qquad \dot{\mu} = 0$$

- If $\dot{\varepsilon} = 0$ and $\dot{X} = V$ then $E_c + V \times B_c = 0$ is the velocity law.

The lagrangian density on extended phase space

- The single particle lagrangian is an invariant (relativity principle) and should be extended to the phase-space coordinates if non canonical coordinates are adopted: if $z^a = (t, X, \gamma)$, $w_a = (P_0, -P, (m/e)\mu)$ and $\tilde{g}$ is the metric tensor in the 5D extended phase-space, then

$$L = -w_a z'^a = -\tilde{g}_{ab} w^b z'^a,$$

  with $a, b = 0, 1, 2, 3, 4$ and summation convention.

- The problem is that in the former eq. both $\tilde{g}$ and w^b are unknown.

- The lagrangian density on extended phase-space is the following:

$$\ell a = f_m L - \frac{\tilde{\mathcal{R}}}{16\pi\tilde{G}},$$

  being f_m the (single particle) distrib. of matter on the extended phase-space, $\tilde{\mathcal{R}}$ and $\tilde{G}$, the scalar curvature and the gravitational constant of the 5D extended phase-space, respectively..

- From ℓa, it'll be addressed
  - ▷ i) the motion of particles on extended phase-space (Lorentz and gravitational forces).
  - ▷ ii) the dynamics of fields (Maxwell's and Einstein's equations).
  - ▷ iii) the collisionless Boltzmann equation.

Einstein's equation on extended phase space

- The relativistic guiding center transformation in a flat phase-space is the translation of time, positions and velocities by quantities that depend on $\gamma \in S^1$:

$$x^\alpha = X^\alpha + \rho^\alpha(\gamma) \tag{1}$$

$$u = U + \nu(\gamma). \tag{2}$$

- The misleading symmetry is the condition: $(m/e)\mu\gamma' = 1 - U^\alpha U_\alpha$.
(it corresponds to $(m/e)\mu\dot{\gamma} = \varepsilon - \mathcal{E}$ in non relativistic regimes)

- The relativistic Lagrangian has the same form after the GC transformation that is a translation:

$$L = -1 - (e/m)A_\alpha(x^\beta)u^\alpha = -1 - (e/m)A_\alpha(X^\beta)U^\alpha,$$

with $\alpha = 0, 1, 2, 3$ and summation convention.

- The form of the Lagrangian remains the same after a translation on phase-space $\Rightarrow$ The extended energy momentum tensor, $\tilde{T}_{ab}$, is the conserved Noether current.

- From the 5d Hilbert Einstein term in the lagrangian density, $\ell a = f_m L - \tilde{R}/(16\pi\tilde{G})$, and from varying the metric tensor,:

$$\tilde{G}_{ab} = 8\pi\tilde{G}\,\tilde{T}_{ab}. \qquad \text{Einstein's equation on the 5D extended phase space} \tag{3}$$

Kaluza Klein metric and unification of forces

- In 5D: $L = -\tilde{g}_{ab} w^a z'^b$, $a, b = 0, 1, 2, 3, 4$ and $\ell a = f_m L - \tilde{R}/(16\pi\tilde{G})$.
- The misleading symmetry: $U^a U_a = 1$.
- The following KK metric tensor is adopted:

$$\tilde{g}_{ab} = \left| \begin{array}{cc} g_{\alpha\beta} - k_G^2 A_\alpha A_\beta & k_G^2 (m/e)^2 \mu A_\alpha \\ k_G^2 (m/e)^2 \mu A_\beta & -k_G^2 (m/e)^4 \mu^2 \end{array} \right| . \tag{4}$$

being $k_G = \sqrt{16\pi G}$.

- being $\sqrt{|\tilde{g}|} = \sqrt{-g}(m/e)^2 k_G \mu$ and $\tilde{G} = G \int (m/e)^2 k_G \mu \tilde{J}_{\mathcal{P}} d\gamma d\varepsilon d\mu$.
- thus,

$$S_{\text{field}} = -\int \frac{\tilde{\mathcal{R}}}{16\pi\tilde{G}} \sqrt{|\tilde{g}|} d^7 z = -\int \sqrt{-g} dt d^3 X \, \frac{R}{16\pi G} - \int \sqrt{-g} dt d^3 X \, \frac{F_{\alpha\beta} F^{\alpha\beta}}{4}. \tag{5}$$

- The terms in the lagrangian density (on the space-time) are the desired ones: e.m. and gravitational, but they are referring to fields on (t, X) where X is the guiding center position and it doesn't indicate the position of a particle $\Rightarrow$ the present theory is NON LOCAL

Boltzmann's and Vlasov's equations without BBGKY

- By writing $L = \mathcal{S}'$, as the proper time derivative of the principal Hamilton function, $\mathcal{S}$, then it is possible to recognize a Legendre's transformation in

$$\ell a = f_m \mathcal{S}' - \frac{\tilde{\mathcal{R}}(f_m, \mathcal{S})}{16\pi \tilde{G}}. \quad \text{like for} \quad L(q, \dot{q}) = p\dot{q} - H(p, q) \tag{6}$$

- The distribution of matter, f_m, is the conjugate momentum of the principal Hamilton function.

- By varying such lagrangian density on 5D extended phase-space with respect to $\mathcal{S}$ and $\mathcal{S}'$, then the Euler-Lagrangian equation is

$$f_m' = -\frac{\partial_{\mathcal{S}} \tilde{\mathcal{R}}}{16\pi \tilde{G}}. \tag{7}$$

- The former is a Boltzmann-like equation, describing the evolution of the distribution function, with a RHS that depends on the geometry instead of collisions.

- If $\tilde{\mathcal{R}}$ doesn't depend on $\mathcal{S}$ then the Vlasov equation is obtained (describing the equilibrium).

- BBGKY is overcame because there aren't collisions but geodesics, the two body interaction could be the effect of a weak field approximation that may not be applicable in a strong magnetized plasma. In such case the fluid picture is obtained without the need of truncations.

Conclusions

- The density lagrangian on the extended phase-space has been proposed:

$$\ell a = f_m L - \frac{\tilde{\mathcal{R}}}{16\pi\tilde{G}},$$

for deriving

  - Lorentz's force law by varying the single particle lagrangian with respect positions and velocities.
  - Maxwell's and Einstein's equations by varying it with respect to the metric tensor on the extended phase space.
  - (Collisionless) Boltzmann's equation by varying it with respect to the principal Hamilton function and the single particle Lagrangian.

- Particles follow geodesics on extended phase-space (unification of forces).
- The single particle distribution is the conjugate momentum of the principal Hamilton function.
- BBGKY is overcame.
- At the equilibrium the Vlasov's equation is obtained.

Thanks a lot

References:

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- Valvo L. and Di Troia C., 46th EPS Conference on Plasma Physics (2019) contribution, "Hamiltonian Formulation of the Non-perturbative Guiding Centre Equation"
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Backup slides

- The lagrangian for describing electrodynamics (without gravitation) for plasmas is the single particle lagrangian, $L(t,x,v)$, times the distribution function of charges, $f_m(t,x,v)$, added to the e.m. lagrangian. The action is often expressed like:

$$S_{\text{plasma}} = \int d\,t d^3\,x d^3\,v f(t,x,v) L(t,x,v) - \int \frac{F_{\alpha\beta}F^{\alpha\beta}}{4} d\,t d^3\,x, \tag{8}$$

- There is an unbalance in the action (8) between the matter action and the field one.

- In principle, for restoring the symmetry between the two lagrangians, it should be simple to think at an action written as

$$S_{\text{plasma}} = \int d\,t d^3\,x d^3\,v \mathcal{L}_{\text{plasma}}, \tag{9}$$

where $\mathcal{L}_{\text{plasma}} = f_m(t,x,v)L(t,x,v) + \text{"somethingnew"}$ and the property that

$$\int \text{"somethingnew"}\, d^3\,v = -\frac{F_{\alpha\beta}F^{\alpha\beta}}{4}. \tag{10}$$

Two key-points: General relativity and BBGKY hierarchy

▷ General relativity in 4 steps

- Translational symmetry: if $x^\alpha = X^\alpha + \rho^\alpha$ (flat space time) then the lagrangian density remains the same.
- Noether current: the energy momentum tensor $T^{\alpha\beta}$ is conserved due to the translational symmetry.
- The density lagrangian density describing the gravitational field is the Hilbert-Einstein one
$$\mathcal{L}_{\mathrm{HE}} = -\frac{R}{16\pi G}$$
- By varying the metric tensor, the Einstein's equation is obtained:
$$G_{\alpha\beta} = 8\pi G\, T_{\alpha\beta}. \tag{11}$$

▷ BBGKY for N interacting particles in 2 steps

- If the Hamiltonian is $\mathcal{H} = \sum_{i=1}^N p_i^2/(2m) + \sum_{i=1}^N U(r_i) + \sum_{i<j}^N V(|r_i - r_j|)$,
- then, the single particle distr. func. depends on the 2-particles distr. func. and so on and so forth until the (N-1)-particles distr. func. depends on the N-particles distr. func.
- that means that you cannot know the single-particle distr. func. if you don't already know the N-particles distr. func. For such reason the transport equations are a truncated set of equations, and the fluid picture is an approximation of the kinetic one.

Lorentz's force law solutions of motion

- A charge e with mass m that (non-relativistically) moves in a externally given and static e.m. field is classically described by the system of equations:

$$\dot{x} = v; \tag{12}$$

$$\dot{v} = (e/m)(E + v \times B) \tag{13}$$

within eulerian description $\dot{x} = v(t, x, \alpha)$ with $\dot{\alpha} \cdot \nabla_\alpha v = 0$. Substituting $E = -\partial_t A - \nabla\Phi$ and $B = \nabla \times A$, eq. (13) becomes

$$\partial_t v + \dot{x} \cdot \nabla v + \dot{\alpha} \cdot \nabla_\alpha v = (e/m)(-\partial_t A - \nabla\Phi + v \times \nabla \times A). \tag{14}$$

or

$$\partial_t[v + (e/m)A] + \nabla[v^2/2 + (e/m)\Phi] = v \times \nabla \times [v + (e/m)A]. \tag{15}$$

Equivalent to the velocity law:

$$E_c + v \times B_c = 0, \tag{16}$$

if the canonical e.m. fields are defined as

$$(e/m)E_c = -\partial_t p - \nabla\varepsilon \qquad \text{and} \qquad (e/m)B_c = \nabla \times p \tag{17}$$

The guiding center description of motion

- The guiding center description of motion is not eulerian nor lagrangian.

$$x = X + \rho(\gamma), \tag{18}$$
$$\dot{x} = V + \sigma(\gamma). \tag{19}$$

- $\dot{X} = V(t, X, \mu, \varepsilon)$ with $E_c + V \times B_c = 0$.
- If $\mathcal{E} = V^2/2 + (e/m)\Phi(t, X)$, then the cyclotron frequency, $\dot{\gamma}$, is chosen to be

$$(m/e)\mu\dot{\gamma} = \varepsilon - \mathcal{E} \tag{20}$$

- the guiding center momentum is $P = V + (e/m)A(t, X)$.
- The lagrangian doesn't explicitly depend on the gyro-phase, γ is cyclic.
- The constant of motion for such symmetry is the magnetic moment, μ.

The guiding center Lagrangian

- The single (charged) particle Lagrangian, L, must be gauge independent: if

$$L \to L + \dot{\mathcal{S}}, \tag{21}$$

the equations of motion are the same.

- L must be a scalar: if the index A runs over the extended phase-space dimensions then

$$L = \gamma_A \dot{z}^A \tag{22}$$

▷ Canonical coordinates:

$$z^A = (t, x, v) \qquad \text{and} \qquad \gamma_A = (-\varepsilon, p, 0).$$

▷ Non canonical, guiding center coordinates:

$$Z^A = (t, X, \gamma, \mu, \varepsilon) \qquad \text{and} \qquad \Gamma_A = (-\mathcal{E}, P, -(m/e)\mu, 0, 0).$$

- $L = p \cdot \dot{x} - \varepsilon = P \cdot \dot{X} - \mathcal{E} - (m/e)\mu\dot{\gamma} \Longleftrightarrow p \cdot \dot{x} = P \cdot \dot{X}$ if $\varepsilon = \mathcal{E} + (m/e)\mu\dot{\gamma}$.
- If $\mathcal{S} = (m/e)\mu\gamma$ then guiding center Lagrangian is

$$L_{gc} = L + \dot{\mathcal{S}} = P \cdot \dot{X} - \varepsilon + (m/e)\mu\dot{\gamma}.$$