Asymptotic preserving methods for plasma simulations

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Outline

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Asymptotic Preserving (AP) Schemes

Framework overview

- Asymptotic parameter : ε defining the regime transition.
- P^{ε} valid for all ε-values but numerically impracticable for $\varepsilon \ll 1$.
- P^{0} valid for $\varepsilon \ll 1$ but meaningless when $\varepsilon = \mathcal{O}(1)$;
  P^{0} is the singular limit of P^{ε} when $\varepsilon \to 0$.

Goals

- Consistancy with P^{ε} for $\varepsilon = \mathcal{O}(1)$, (small scales resolved by the mesh),
- Consistancy with P^{0} for $\varepsilon \ll 1$, (mesh under-resolved),
- Switch from one regime to another according to ε values,
- Uniform stability / ε.

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AP-Schemes

Asymptotic preserving schemes

Introduced by Shi Jin for multiscale kinetic equations
[S. Jin] *Efficient Asymptotic-Preserving (AP) schemes for some multiscale kinetic equations*, Siam J. Sci. Comp., 21 (1999) pp 441.

Main concepts

- AP-Schemes : what for ?
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  - Consistancy with the limit model P^0 when $\varepsilon \to 0$
  - Uniformly stable / asymptotic parameter ε
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Quasi-neutral limit of the Vlasov-Maxwell system

Dimensionless V-M system

$$\partial_t f + v \cdot \nabla_x f = (E + v \times B) \cdot \nabla_v f \,, \qquad \text{(V)}$$
$$-\varepsilon \partial_t E + \nabla \times B = J \,, \qquad \text{(A)}$$
$$\partial_t B + \nabla \times E = 0 \,, \qquad \text{(F)}$$
$$\varepsilon \nabla \cdot E = 1 - n \,, \qquad \text{(G)}$$
$$\nabla \cdot B = 0 \,.$$

Vlasov eq. moments

$$\partial_t n - \nabla \cdot J = 0 \qquad \text{(M0)}$$
$$\partial_t J = \nabla \cdot S + nE - J \times B \qquad \text{(M1)}$$

with $\qquad S = \int (v \otimes v) \, f \, dv \,.$

Electric field computation in the quasi-neutral regime

- The time derivative of (A) together with the curl of (F) yields

$$\nabla \times \nabla \times E = -\partial_t J \,,$$

- inserting the eq. (M1) allows for the entire field computation

$$n\, E + \nabla \times \nabla \times E = J \times B - \nabla \cdot S$$

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F. Chen, **Introduction to plasma physics and controlled fusion** (1984) : « *Do not use Poisson's equation to obtain E unless it is unavoidable !* »

« *In a plasma it is usually possible to assume $n_i = n_e$ and $\nabla \cdot E \neq 0$ at the same time. We shall call this the plasma approximation.* »
In high-frequency phenomena « *the plasma approximation is not valid and E must be found from Maxwell's equation rather than from the electron and ion equations of motion.* » AP schemes are an implementation of this statement.

Drift limits (Gyro-fluid limit)

Dimensionless (E-L) model

$$\frac{\partial n}{\partial t} + \nabla \cdot q = 0\,,$$

$$\frac{\partial q}{\partial t} + \nabla \cdot \left(q \otimes \frac{q}{n}\right) + \frac{1}{\varepsilon}\nabla p = \frac{1}{\varepsilon}\left(nE + q \times B\right)\,,$$

$$\frac{\partial W}{\partial t} + \nabla \cdot ((W+p)\frac{q}{n})) = E \cdot q\,,$$

$$W = \varepsilon \frac{1}{2}\frac{q^2}{n} + \frac{3p}{2}\,, \qquad p = nT\,.$$

Drift regime $\varepsilon \to 0$ (gyrofluide)

- Perpendicular to B drift velocities $u_\perp = (-E \times B + (\nabla p)/n \times B)|B|^2\,.$
- Along B the pressure waves travel at infinite speed to balance the electric forces

$$\nabla_\parallel p = nE_\parallel\,. \qquad (M_\parallel)$$

In this equation $q_\parallel$ has vanished.

Reformulation : cure the degeneracy of the parallel momentum equation

"Low-Mach" approach : compute the pressure in order to prevent the degeneracy of the parallel momentum equation.

$$\left\{\begin{array}{l} \dfrac{\partial^2 p}{\partial t^2} - \nabla \cdot \left(\left((\mathbb{I}\mathrm{d} - b \otimes b) + \dfrac{1}{\varepsilon}(b \otimes b)\right)(\nabla p - nE)\right) = f\,,\ \ b = B/|B|\,, \\ \text{Periodic boundary conditions on } \Gamma_\parallel\,. \end{array}\right.$$

Outline

Problem statement

Model problem : Anisotropic diffusion equations

$$\frac{\partial^2 p}{\partial t^2} - \nabla \cdot \left(\frac{1}{\varepsilon}(b \otimes b)\nabla p + (\mathbb{I}\mathrm{d} - b \otimes b)\nabla p \right) = f^{\varepsilon} ,$$

with periodic boundary conditions along the magnetic field lines.

Toy Model

$$(P^{\varepsilon}) \begin{cases} -\partial_x(A\,\partial_x \phi^{\varepsilon}) - {}^{1}/_{\varepsilon}\,\partial_z(A_z\,\partial_z\phi^{\varepsilon}) = f \ \text{ in } \Omega\,, & (*) \\ \phi = 0 \ \text{ on } \partial\Omega_x\,, \quad \partial_z\phi^{\varepsilon} = 0 \ \text{ on } \partial\Omega_z\,. \end{cases}$$

What can we expect from standard numerical methods ?

- For $\varepsilon \ll 1$ consistency with the degenerate problem

$$(\tilde{P}^0) \begin{cases} -\partial_z(A_z\partial_z\phi^0) = 0 \text{ in } \Omega \\ \partial_z\phi^0 = 0 \text{ on } \partial\Omega_z. \end{cases}$$

- Blow-up of the condition number with $\varepsilon \to 0$.

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- ϕ^ε verifies $-\partial_x(\overline{A\,\partial_x\phi^\varepsilon}) = \bar{f}$ in Ω, for $\varepsilon \geq 0$, with : $\bar{f} = {}^1\!/_{L_z} \int_0^{L_z} f(x,z)dz$.
- $(\tilde{P}^0)$ provides $\partial_z \phi^0 = 0$

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Limit regime : Singular limit

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Derivation of the AP-Scheme

« Macro-Micro » decomposition

$$\phi(x,z) = \bar{\phi}(x) + \phi'(x,z)\,,$$

$$\bar{\phi} = \frac{1}{L_z}\int \phi(x,z)\,dz\,, \qquad \phi'(x,z) = \phi(x,z) - \bar{\phi}(x)\,, \quad \bar{\phi}' = 0\,,$$

Reformulated problem

1/2-D elliptic problem for $\bar{\phi}$

$$\partial_x(\bar{A}_x\partial_x\bar{\phi}) = \bar{f} + \partial_x\left(\overline{A_x'\partial_x\phi'}\right)\,, \quad \bar{\phi} = 0 \text{ on } \partial\Omega_x\,,$$

2/3-D elliptic (well posed) problem for ϕ'

$$\begin{cases} -\partial_z(A_z\partial_z\phi') - \varepsilon\partial_x(A_x\partial_x\phi') + \varepsilon\partial_x\left(\overline{A_x\partial_x\phi}\right) = \varepsilon f - \varepsilon\bar{\phi}\,, \\ \partial_z\phi' = 0 \quad \text{on} \quad \Omega_x\times\partial\Omega_z\,, \qquad \phi' = 0 \quad \text{on} \quad \partial\Omega_x\times\Omega_z\,, \\ \overline{\phi'} = 0\,, \quad \text{in} \quad \Omega_x\,. \end{cases}$$

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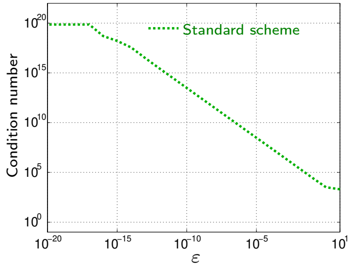

Condition number as a function of ε.

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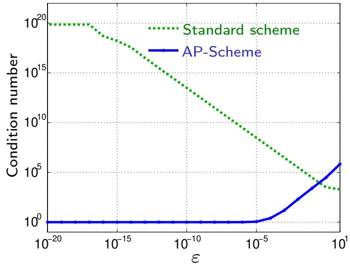

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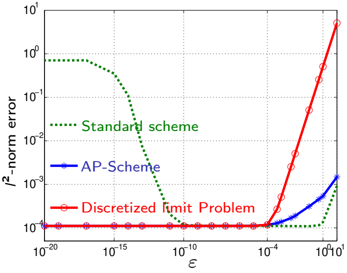
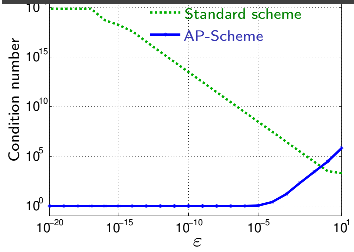


Error between the exact solution and the numerical approximations as a function of ε



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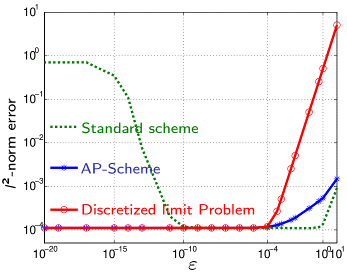

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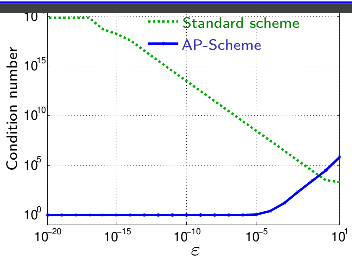

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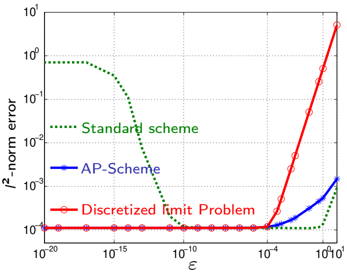

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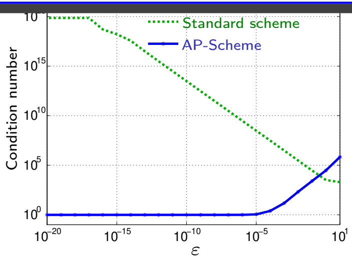

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Generalization to non aligned coordinates

« Duality-Based » formulation

$$\phi(x,z) = \bar{\phi}(x) + \phi'(x,z) \text{ with } < \phi, \phi' > := \int_{\Omega_z} \bar{\phi}\phi' dz = 0 .$$

« Micro-Macro » formulation

$$\phi(x,z) = p(x) + Q(x,z) \text{ with } Q = 0 \text{ on } \Gamma_{in} := \{x \in \partial\Omega,\ b(x)\cdot n(x) > 0\} .$$

$$(iMM) \begin{cases} -\partial_x(A_x \partial_x \phi^\varepsilon) - \partial_z(A_Z \partial_z Q^\varepsilon) = f \text{ in } \Omega , \\ \phi = 0 \text{ on } \partial\Omega_x , \quad \partial_z \phi^\varepsilon = 0 \text{ on } \partial\Omega_z , \\ -\partial_z(A_Z \partial_z \phi^\varepsilon) = -\varepsilon\partial_z(A_Z \partial_z Q^\varepsilon) \text{ in } \Omega , \\ Q^\varepsilon(x,0) = 0 , \\ \partial_z Q^\varepsilon = 0 \text{ on } \partial\Omega_z \end{cases}$$

Closed field lines

Stabilized « Micro-Macro » formulation

Harness the truncature error of the discretization to ensure uniqueness of Q^ε :

$$(sMM)\begin{cases} -\partial_x(A_x\,\partial_x\phi^\varepsilon) - \partial_z(A_Z\,\partial_z Q^\varepsilon) = f \ \text{ in } \Omega\,, \\ \phi = 0 \ \text{ on } \partial\Omega_x\,, \quad \partial_z\phi^\varepsilon = 0 \ \text{ on } \partial\Omega_z\,, \\ -\partial_z(A_Z\,\partial_z\phi^\varepsilon) = -\varepsilon\partial_z(A_Z\,\partial_z Q^\varepsilon) + \sigma h^2 Q^\varepsilon\,, \ \text{ in } \Omega\,, \\ \partial_z Q^\varepsilon = 0 \ \text{ on } \partial\Omega_z \end{cases}$$

σ numerical parameter (stabilization)

« Saddle Point » problems : Augmented system

$$\begin{pmatrix} -\Delta & -(1-\varepsilon)\Delta_\| \\ -\Delta_\| & \varepsilon\Delta_\| + \sigma h^2 M \end{pmatrix}$$

- Δ : Laplacian, $\Delta_\|$: Parallel Laplacian, M : mass matrix ;
- Matrix Conditioning $\sim 1/h^4$ (iMM), $\sim 1/(\sigma h^2)$ (sMM)

TFI : Compute the sequence $(\phi^n, q^n)_{n>0}$

For $\varepsilon_0 > 0$ we define the mildly anisotropic operator

$$\Delta_{\varepsilon_0} u := \varepsilon_0 \Delta_\perp u + \Delta_{||} u \,, \tag{1}$$

with

$$\Delta_{||} u := \nabla \cdot \left(A_{||} (b \otimes b) \nabla u \right) \,, \quad \Delta_\perp u := \nabla \cdot \left(A_\perp (\mathbb{I}\mathrm{d} - b \otimes b) \nabla u \right) \,,$$
$$\nabla_{||} u := (b \otimes b) \nabla u \,, \quad \nabla_\perp := (\mathbb{I}\mathrm{d} - b \otimes b) \nabla u \,,$$
$$\mathbb{A}_{\varepsilon_0} := \varepsilon_0 A_\perp (\mathbb{I}\mathrm{d} - b \otimes b) + A_{||} (b \otimes b) \,.$$

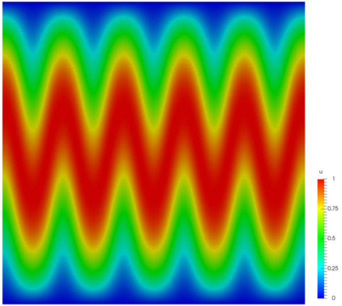
$$\begin{cases} -\Delta_{\varepsilon_0} \phi^{n+1} = \varepsilon_0 f + (\varepsilon_0 - \varepsilon) \Delta_{||} q^n & \text{in } \Omega, \\ n \cdot \mathbb{A}_{\varepsilon_0} \nabla \phi^{n+1} = -(\varepsilon_0 - \varepsilon) n \cdot \left(A_{||} \nabla_{||} q^n \right) & \text{on } \Gamma_N \,, \\ \phi^{n+1} = 0 & \text{on } \Gamma_D \,, \end{cases} \tag{2}$$

$$\begin{cases} -\Delta_{\varepsilon_0} q^{n+1} = f + \Delta_\perp (\phi^{n+1} - \varepsilon_0 q^n) & \text{in } \Omega, \\ n \cdot \mathbb{A}_{\varepsilon_0} \nabla q^{n+1} = -n \cdot \left(A_\perp \nabla_\perp \left(\phi^{n+1} - \varepsilon_0 q^n \right) \right) & \text{on } \Gamma_N \,, \\ q^{n+1} = 0 & \text{on } \Gamma_D \,, \end{cases} \tag{3}$$

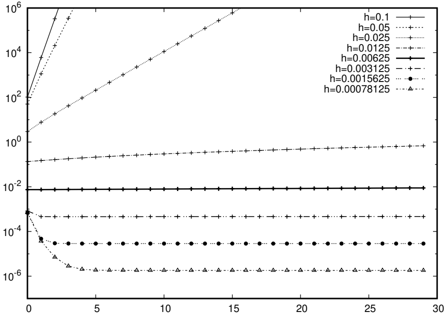
Open field lines with oscillating directions

Magnetic field definition

$$b = \frac{B}{|B|}, \quad B = \begin{pmatrix} \alpha(2y-1)\cos(m\pi x) + \pi \\ \pi\alpha m(y^2 - y)\sin(m\pi x) \end{pmatrix}, \tag{4}$$



(a) $\alpha = 2$, $m = 10$



(b) TFI convergence

Figure 1 – Exact solution ($\alpha = 2, m = 10$) and error approximation as a function of the TFI fixed point inetrations : $\mathbb{Q}_2$-FEM, Cartesian mesh, $\varepsilon = 10^{-15}$.

Diffusion in a ring : $\mathbb{P}_1$-FEM, unstructured Mesh, $\varepsilon = 10^{-15}$.

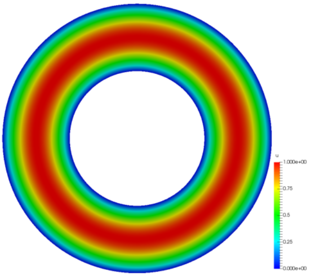
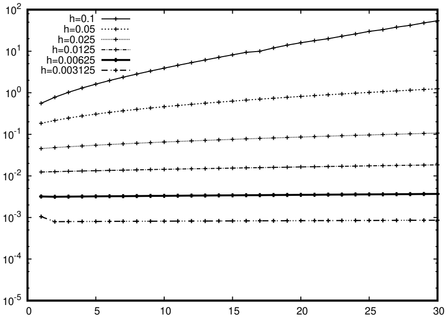


Figure 2 – Exact solution.



(a) $\varepsilon_0 = 10^{-3}$

Figure 3 – Error $\|\phi - \phi^h\|_2$ as a function of the fixed point iteration.

Outline

Gyro-fluid limit : A toy problem

The dimensionless Euler-Lorentz system

$$
\begin{aligned}
&\frac{\partial n}{\partial t} + \nabla \cdot q = 0\,, \qquad q := nu\,, \\
&\frac{\partial q}{\partial t} + \nabla \cdot \left(q \otimes \frac{q}{n} \right) + \frac{1}{\varepsilon} \nabla p = \frac{1}{\varepsilon} \left(nE + q \times B \right)\,, \\
&\frac{\partial W}{\partial t} + \nabla \cdot \left((W + p) \frac{q}{n}) \right) = E \cdot q\,, \\
&W = \varepsilon \frac{1}{2} \frac{q^2}{n} + \frac{3p}{2}\,, \qquad p = nT\,.
\end{aligned}
$$

Interpretation of the asymptotic parameter

- ε may be defined as the Mach number and the re-scaled cyclotron period
- or equivalently as the dimensionless ion gyro-radius (with respect to the thermal velocity) $\varepsilon = \rho_*^2$.

Drift regime $\varepsilon \to 0$ (gyrofluide)

- Perpendicular to B drift velocities $u_\perp = (-E \times B + (\nabla p)/n \times B)|B|^2$.
- Along B the pressure waves travel at infinite speed to balance the electric forces

$$
\nabla_\| p = nE_\|\,. \qquad (M_\|)
$$

The parallel momentum balance is degenerate in the drift limit (no eq. providing $u_\|$).

The dimensionless Euler-Lorentz system

$$\frac{\partial n}{\partial t} + \nabla \cdot q = 0\,,$$
$$\frac{\partial q}{\partial t} + \nabla \cdot \left(q \otimes \frac{q}{n}\right) + \frac{1}{\varepsilon}\nabla p = \frac{1}{\varepsilon}\left(nE + q \times B\right)\,,$$
$$\frac{\partial W}{\partial t} + \nabla \cdot ((W + p)\frac{q}{n})) = E \cdot q\,,$$
$$W = \varepsilon \frac{1}{2}\frac{q^2}{n} + \frac{3p}{2}\,, \qquad p = nT\,.$$

Drift regime $\varepsilon \to 0$ (gyrofluide)

- Perpendicular to B drift velocities $u_\perp = (-E \times B + (\nabla p)/n \times B)|B|^2\,.$
- Along B the pressure waves travel at infinite speed to balance the electric forces

$$\nabla_\parallel p = nE_\parallel\,. \qquad (M_\parallel)$$

In this equation $q_\parallel$ has vanished.

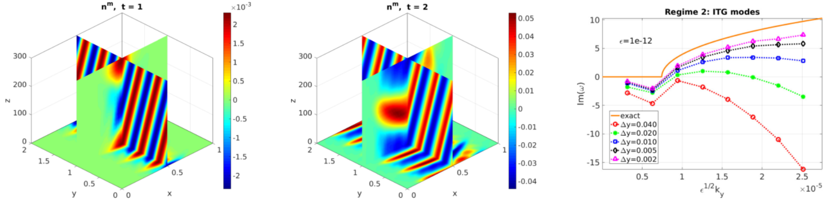
Reformulation : cure the degeneracy of the parallel momentum equation

① Harness $q_\parallel$ as the Lagrange multiplier of the parallel balance $(M_\parallel)$.

② "Low-Mach" approach : compute the pressure in order to prevent the degeneracy of the parallel momentum equation.

$$\begin{cases} \dfrac{\partial^2 p}{\partial t^2} - \nabla \cdot \left(\left((\mathbb{I}\mathrm{d} - b \otimes b) + \dfrac{1}{\varepsilon}(b \otimes b) \right)(\nabla p - nE) \right) = f\,,\ \ b = B/|B|\,, \\ \text{Periodic boundary conditions on } \Gamma_\parallel\,. \end{cases}$$

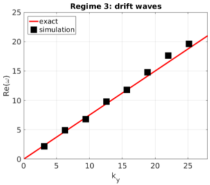
3D slab ITG Simulation by the Drift-Asymptotic Scheme



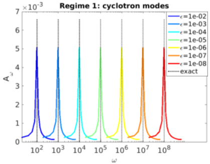
(a) Density at t=1. (b) Density at t=2. (c) Inst. growth rate.

Figure 4 – Drift-Asymptotic scheme simulations : intermediate regime (ITG).

3D slab ITG Simulation by the Drift-Asymptotic Scheme



(a) Drift waves.



(b) Cyclotron waves.

Figure 5 – The scheme allows for the simulations of the low and the fast frequencies (no approximation in the equations).

A « gyro-fluid » model without the drift-expansions.

Gyro-fluid models

- Derived analytically by expanding the equations according to a small parameter (dimensionless cyclotron period, gyro-radius, ...) ;
- High frequencies are filtered out analytically from the equations ;
- Large number of terms to enlarge the validity range, preserve the (energy) conservation properties : complex set of equations.

Drift-Asymptotic numerical method

- Work on the primitive (fluid) equations ;
- Perform numerically the drift approximation into the primitive equations ;
- Simple set of equations (conservative form) in a singular limit : Asymptotic-Preserving methods.

Outline

The quasineutral Vlasov Maxwell system P^0

This set of equations is stated as the limit problem

$$\begin{aligned}
&\partial_t f + v \cdot \nabla_x f - (E + v \times B) \cdot \nabla_v f = 0\,, \\
&nE + \nabla \times \nabla \times E = J \times B - \nabla \cdot S\,, \\
&\partial_t B + \nabla \times E = 0\,, \\
&\nabla \cdot B = 0\,.
\end{aligned}$$

The quasineutral asymptotic is a Singular limit

- ☞ The nature of the equation providing the electric field is not the same in P^λ and P^0.
- ☞ Reformulation of the equations : provide a set of equations containing both regimes with a smooth transition accordingly to λ.
- ☞ Perform the same derivation on the Vlasov-Maxwell system with $\lambda > 0$.

The quasineutral Vlasov Maxwell system P^0

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&\partial_t B + \nabla \times E = 0\,,\\
&\nabla \cdot B = 0\,.
\end{aligned}
$$

$$
\begin{tikzcd}
P^{\lambda,h} \arrow[r, "h \to 0"] \arrow[d, "\lambda \to 0"] & P^{\lambda} \arrow[d, "\lambda \to 0"] \\
P^{0,h} \arrow[r, "h \to 0"] & P^{0}
\end{tikzcd}
$$

AP-schemes consistancy properties.

The quasineutral asymptotic is a Singular limit

- The nature of the equation providing the electric field is not the same in P^λ and P^0.
- Reformulation of the equations : provide a set of equations containing both regimes with a smooth transition accordingly to λ.
- Perform the same derivation on the Vlasov-Maxwell system with $\lambda > 0$.

The Reformulated Vlasov-Maxwell system

Set of equations equivalent to the Vlasov-Maxwell system

$$
\begin{aligned}
&\partial_t f + v \cdot \nabla_x f - (E + v \times B) \cdot \nabla_v f = 0\,, \\
&\lambda^2 \partial_{tt}^2 E + n\, E + \nabla \times \nabla \times E = J \times B - \nabla \cdot S\,, \\
&\partial_t B + \nabla \times E = 0\,, \\
&\nabla \cdot B = 0\,.
\end{aligned}
$$

Derivation of the reformulated equation

$$
\begin{aligned}
&\partial_t(-\lambda^2 \partial_t E + \nabla \times B) = \partial_t J\,, \\
&\nabla \times (\partial_t B + \nabla \times E) = 0\,, \\
&\partial_t J = \nabla \cdot S + nE - J \times B\,,
\end{aligned}
$$

The Reformulated Vlasov-Maxwell system

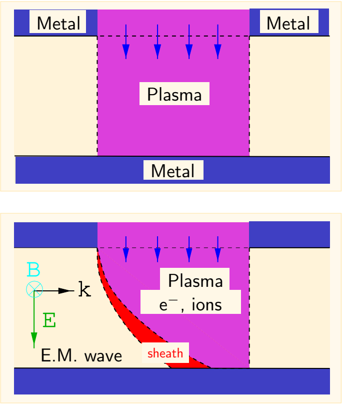
Set of equations equivalent to the Vlasov-Maxwell system

$$\begin{aligned}
&\partial_t f + v \cdot \nabla_x f - (E + v \times B) \cdot \nabla_v f = 0\,, \\
&\lambda^2 \partial_{tt}^2 E + n\, E + \nabla \times \nabla \times E = J \times B - \nabla \cdot S\,, \\
&\partial_t B + \nabla \times E = 0\,, \\
&\nabla \cdot B = 0\,.
\end{aligned}$$

Consistancy with the limit problem

$$\begin{aligned}
&\partial_t f + v \cdot \nabla_x f - (E + v \times B) \cdot \nabla_v f = 0\,, \\
&n\, E + \nabla \times \nabla \times E = J \times B - \nabla \cdot S\,, \\
&\partial_t B + \nabla \times E = 0\,, \\
&\nabla \cdot B = 0\,.
\end{aligned}$$

Plasma Opening switch simulation



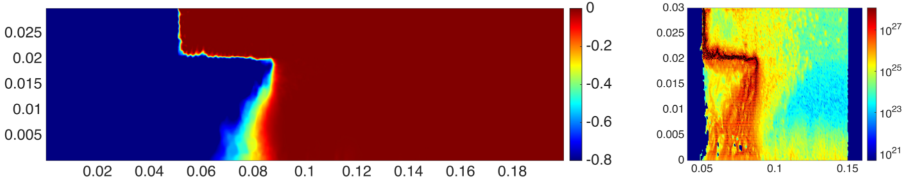
POS schematic representation

Test case description

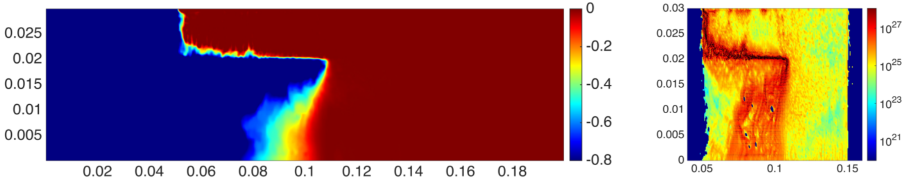
- One dimensional geometry E_x, E_y, B_z, $q_{\alpha,x}$, $q_{\alpha,y}$ and n_α are functions of x.
- Plasma density : $10^{16} - 10^{18}$ m^{-3}.
- Incident EM-wave : $E_y = -10^8$V/m, rising time 10 ns.

Schemes comparison

- Reference : standard (exp./imp) scheme with $\Delta x < \lambda_D$ and $\Delta t < \omega_p^{-1}$.
- Low density POS : $\Delta x \approx \lambda_D$
- High density POS : $\Delta x \approx 10^2 \lambda_D$

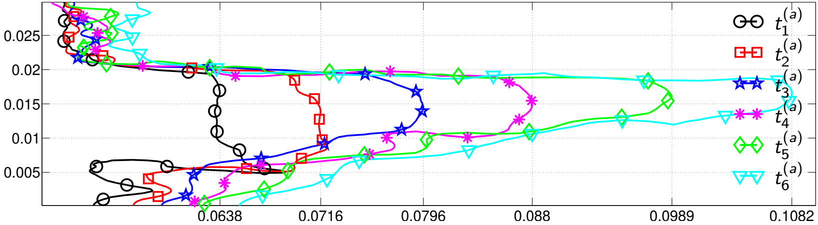


(a) Magnetic field B_z and electronic momentum at time t_4.



(b) Magnetic field B_z and electronic momentum at time t_6.

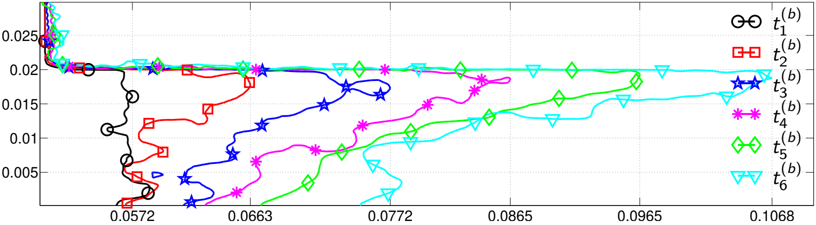
Figure 6 – KMC wave propagation : magnetic field (Tesla) and electronic momentum $(m^{-2} \cdot s^{-1})$ as functions of the space coordinates at time $t_4 = 2.58$ns and $t_6 = 3.67$ns computed with the set-up (b) detailed in Figure 7.



(a) Magnetic field level set as a function of the x and y space variables at times $t_1^{(a)} = 1.75$, $t_2^{(a)} = 2.16$, $t_3^{(a)} = 2.56$, $t_4^{(a)} = 3.03$, $t_5^{(a)} = 3.56$ and $t_6^{(a)} = 4.09$ ns (set-up (a)).

Set-up	Density	T_e	$\bar{\lambda} = \lambda/(\Delta x, \Delta y)$	V_{1-2}	V_{2-3}	V_{3-4}	V_{4-5}	V_{5-6}
	m^{-3}	K		$\times 10^6$ m · s^{-1}				
(a)	$10^{19} - 10^{20}$	$6\ 10^4$	$8.3\,10^{-4} - 5.5\,10^{-3}$	19.4	19.8	17.8	20.7	17.3
(b)	$10^{19} - 10^{21}$	$6\ 10^2$	$1.1\,10^{-4} - 1.8\,10^{-4}$	24.3	24.8	17.9	18.8	18.2

Figure 7 – KMC wave velocity ($2.5\ 10^7$) estimated (V_{i-j}) thanks to the evolution of the magnetic field level set ($B_z = -0.8$ Tesla) on the time interval $[t_i, t_j]$. Set-up (a) (resp. (b)) : grid with 100×100 (resp. 400×100) cells and 10^5 (resp. $4\ 10^6$) particles.



(b) Magnetic field level set as a function of the x and y space variables at times $t_1^{(b)} = 1.24$, $t_2^{(b)} = 1.62$, $t_3^{(b)} = 2.06$, $t_4^{(b)} = 2.58$, $t_5^{(b)} = 3.11$ and $t_6^{(b)} = 3.67$ ns (set-up (b)).

Set-up	Density	T_e	$\bar{\lambda} = \lambda/(\Delta x, \Delta y)$	V_{1-2}	V_{2-3}	V_{3-4}	V_{4-5}	V_{5-6}
	m^{-3}	K		$\times 10^6$ m $\cdot$ s^{-1}				
(a)	$10^{19} - 10^{20}$	$6\ 10^4$	$8.3\,10^{-4} - 5.5\,10^{-3}$	19.4	19.8	17.8	20.7	17.3
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Conclusions

Asymptotic-Preserving methods in few words

- Overcome singular limits.
- Bridging different asymptotic models with a single numerical method (no interface).

Other limits and applications

- Bridging Valsov-Poisson to the Boltzmann relation (Fluid, Quasi-neutral and Drift limits)
- Shear ITG (Gyro-fluid)
- ...