Hybrid formulation of fully kinetic and gyrokinetic Hamiltonian field theory for astrophysical plasmas

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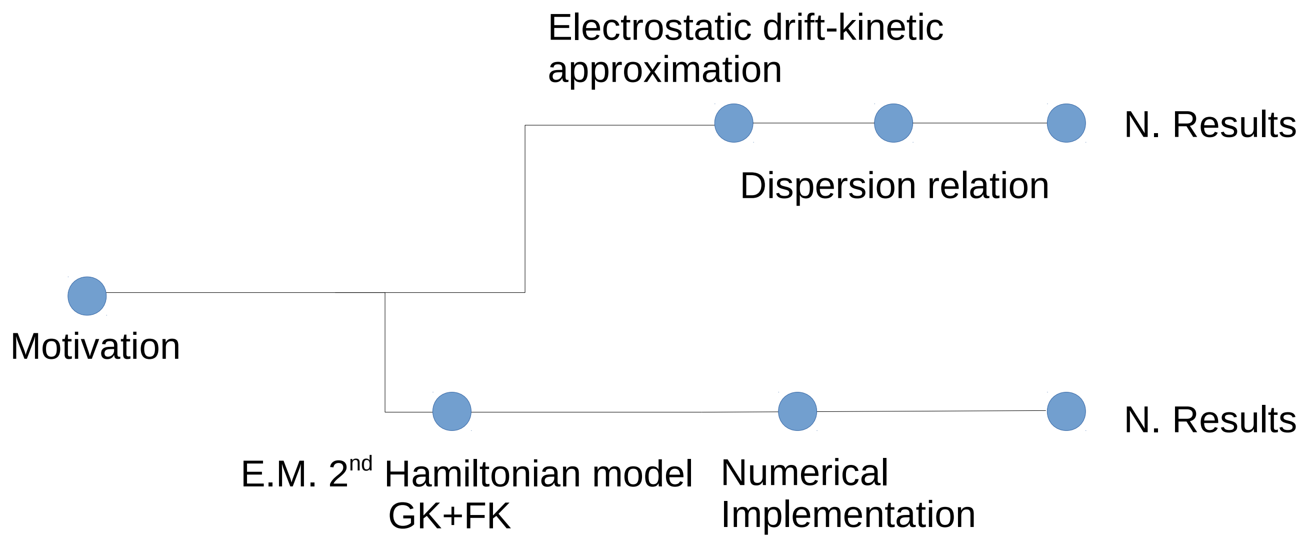
Work-flow


Electrostatic drift-kinetic
approximation
N. Results
Dispersion relation
Motivation
N. Results
E.M. 2nd Hamiltonian model
GK+FK
Numerical
Implementation

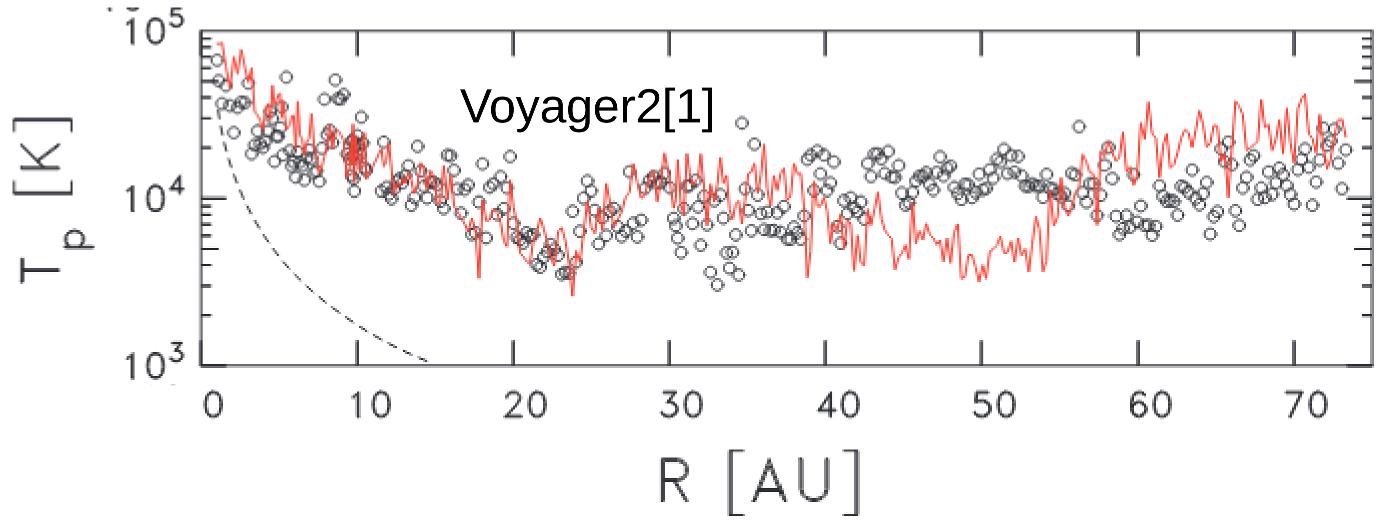
- Work Flow

- Motivation

- Electromagnetic 2nd order GK-FK Hamiltonian theoretical framework

Space Physics Motivation

- Heating in Solar wind (invaluable *in situ* data)



  - Adiabatic approximation fails → Turbulent heating

[1]The Astrophysical Journal, 638:508–517, 2006 February 10

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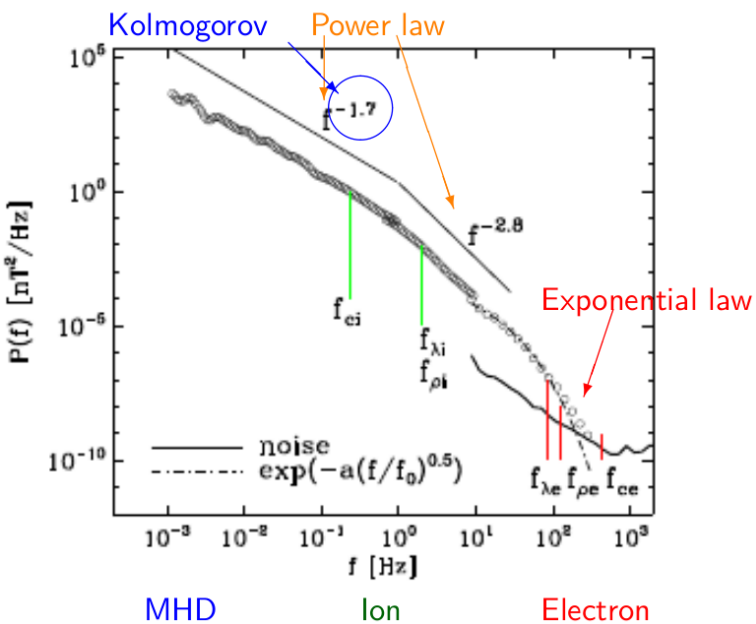
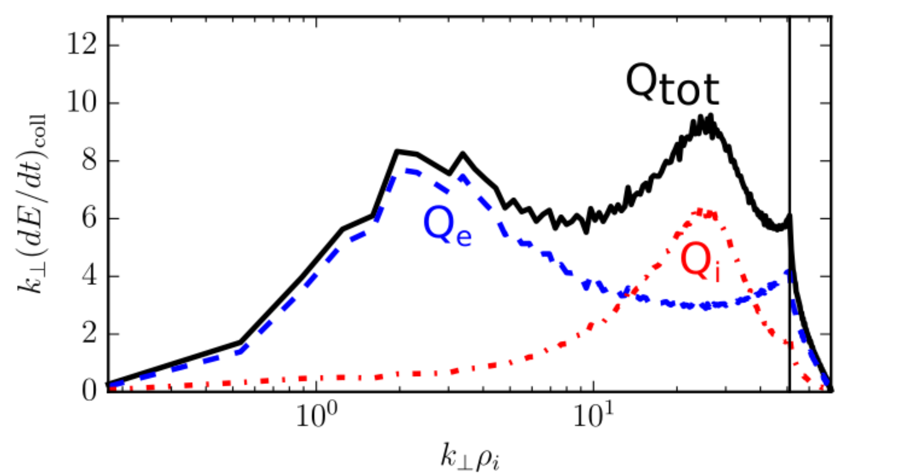
Space Physics Motivation

Magnetic spectrum in the Solar wind



*Distribution of **collisional** dissipation[2]*



For e- heating in ion scale → Landau damping

Electron kinetics is significant in astrophysical environment

[2]Phys. Rev. Lett. 115, 025003

Beyond solar wind

- Ion-to-electron thermal disequilibrium in accretion disks

  - "*Within this scenario, the relative amount of electron heating in the low-β_i, central region of the disk turns out to be crucial to enable a detectable jet.*"*[3][4]*

  - Electron landau damping better described in the context of gyrokinetic if compared to electron-fluid models.

[3] PNAS January 15, 2019 116 (3) 771-776
[4] Mon Not R Astron Soc 478:5209–5229

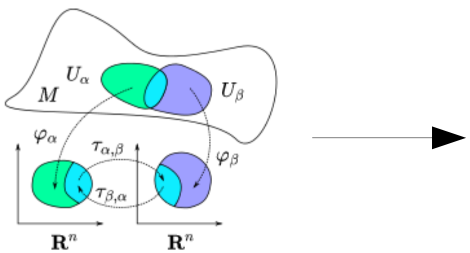
Mathematical motivation

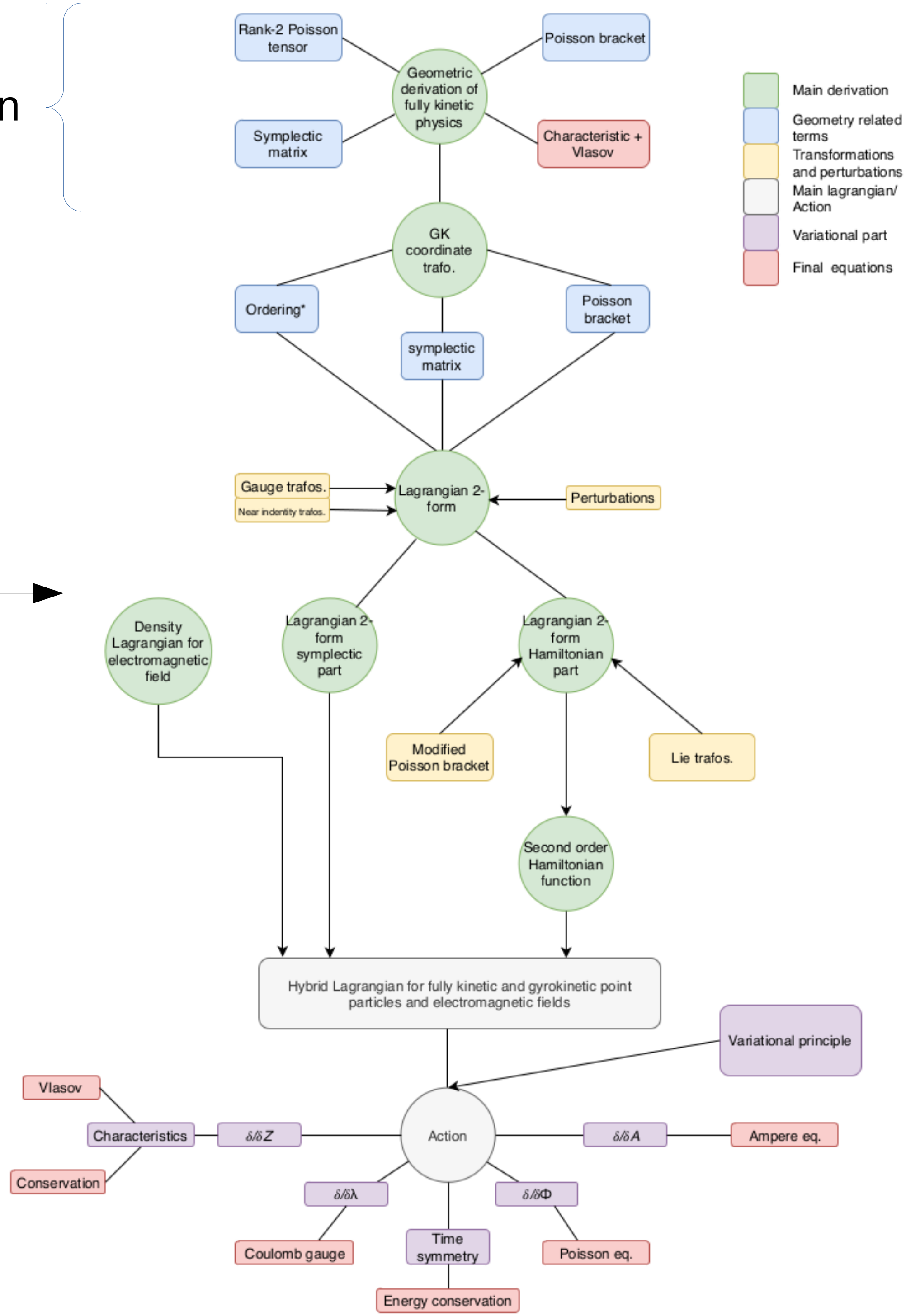
- Consistency[5]
  - Yields a model without *ad hoc* constituents → completeness.
  - Hamiltonian structures preserved up to numerical implementation (important for e. g. energy preserving).
  - More reliable interaction between different species through field equations (GK vs. FK).

“No new calculations are reported here. The new formalism is applied to standard problems such as plasma oscillations and hydrodynamic waves, leads to the same results as the old ones with approximately the same degree of algebraic complication. It is of course, hoped that the Lagrangian method may be powerful enough to produce a useful new way of attacking practical problems.”[5]

[5]*A Lagrangian formulation of the Boltzmann-Vlasov equation for plasmas* Proceedings of the Royal Society of London. Series A. Mathematical and Physical Sciences F. E. Low 1958

Fully kinetic derivation





Gyrokinetic coordinate transformation

Vlasov + field equations

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Geometric formulation of fully kinetic physics

Construction of symplectic matrix from the Lagrangian 1-form

$$\omega_{\alpha\beta} = \epsilon_{ijk} B_k dx^i \wedge dx^j - m\delta_{ij} dx^i \wedge dv^j + m\delta_{ij} dv^i \wedge dx^j$$

The rank-2 Poisson tensor is constructed from the symplectic matrix

$$\Pi^{\alpha\beta} = \omega_{\alpha\beta}^{-1} = \begin{pmatrix} 0 & \frac{1}{m}\delta^{ij} \\ -\frac{1}{m}\delta^{ij} & \frac{1}{m^2}\epsilon^{ijk} B_k \end{pmatrix}$$

The above procedure allow us to generate a Poisson Bracket, and the structure of the system

$$\Pi(A, K) \equiv \{A, K\} = \sum_{\alpha\beta} \frac{\partial A}{\partial Z^\alpha} \Pi^{\alpha\beta} \frac{\partial K}{\partial Z^\beta} = \frac{1}{m}\left(\nabla A \frac{\partial K}{\partial v} - \frac{\partial A}{\partial v} \nabla K\right) + B\left(\frac{\partial A}{\partial v} \times \frac{\partial K}{\partial v}\right)$$

[6] V. I. Arnold *Mathematical Methods of Classical Mechanics*.
[7] Robert H. Wasserman. *Tensor and Manifolds, with applications to Physics.*

The Equations of Motion can be derived from the variation of the phase space Lagrangian, as performed later, or from the Poisson Bracket

$$\dot{x}^I = \{x^I, H\} = \delta^{ij}\frac{\partial H}{\partial v^j} = \Pi^{x^i x^j}\frac{\partial H}{\partial x^j} + \Pi^{x^i v^j}\frac{\partial H}{\partial v^j} = v^I$$

$$\dot{v}^I = \{v^I, H\} = \Pi^{v^i x^i}\frac{\partial H}{\partial x^j} + \Pi^{v^i v^j}\frac{\partial H}{\partial v^j} = -\delta^{ij}\frac{\partial H}{\partial x^j} + \epsilon^{ijk}\frac{\partial H}{\partial v^i} = \frac{\partial \phi}{\partial x^i} + \epsilon^{ijk} v_j B_k$$

The Vlasov equation for the fully kinetic species can be constructed considering the Liouville theorem and the scale of the Larmor radius in Astrophysical environments, i.e. weakly collisional system.

$$\frac{\partial F^I}{\partial t} + \left\{x^I, H^I\right\}\nabla F^I + \left\{v^I, H^I\right\}\partial_{v^I} F^I = 0.$$

Gyrokinetic coordinate transformation

Firstly we define an ordering for the fluctuations

$$\frac{(k_{\|}\rho_{th,e})e\phi}{T_i} = \epsilon_\delta >> \frac{\phi_{th,e}}{|\nabla B/B|^{-1}}$$

Construction of the symplectic matrix from the Lagrangian 2-form

$$\omega_{\alpha\beta} = \begin{pmatrix} \omega_{x^i x^j} & \omega_{x^i v^j} \\ \omega_{v^i x^j} & \omega_{v^i v^j} \end{pmatrix}$$

This give us also the Euler Lagrange equations, similarly to the fully kinetic process

$$\omega_{\alpha\beta}\frac{dZ^\beta}{dt} = \frac{\partial H}{\partial Z^\alpha}$$

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Guiding Center Transformation

After introducing non uniformities in the magnetic field, one must separate the symplectic part of the Lagrangian in order to remove theta dependency.

$$\Gamma \rightarrow \Gamma_0 + \Gamma_1$$

Because we are working with 2-forms, the addition of gauge transformations in the Lagrangian does not change the dynamics of the system.

$$\sigma_1 = -\sum_{n=1} \frac{1}{n!} \frac{e}{\epsilon c} (\rho_0 \cdot \nabla)^{n-1} A \cdot \rho_0$$

A total of four gauge transformations are performed, together with one near identity coordinate transformation and one velocity shift, in order to remove the theta dependency of the symplectic part of the Lagrangian up to second order on ε_B.

The above transformations leave all the theta dependency up to the chosen ordering in the Hamiltonian part of the phase space Lagrangian.

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Gyrokinetic transformation

The final step consists of a Lie transformation on the perturbed canonical Hamiltonian 0-form

$$H_{gc}(\mathbb{Z}_{gc}) = e^{-\pounds_s} H(\mathbb{Z})$$

Theta dependency is eliminated up to chosen ordering via Lie transformation on Hamiltonian part of the 2-form phase space Lagrangian. Our Poisson Bracket reads:

$$\{\mathcal{A},\mathcal{B}\}_{gy} = \epsilon^{-1}\left(\frac{\partial\mathcal{A}}{\partial\overline{\theta}}\frac{\partial\mathcal{B}}{\partial\overline{\mu}} - \frac{\partial\mathcal{B}}{\partial\overline{\mu}}\frac{\partial\mathcal{A}}{\partial\overline{\theta}}\right) + \\ + \frac{B^*}{B^*_\parallel}\left(\nabla^*\mathcal{A}\frac{\partial\mathcal{B}}{\partial\overline{v}_\parallel} - \frac{\partial\mathcal{A}}{\partial\overline{v}_\parallel}\nabla^*\mathcal{B}\right) - \\ -\epsilon\frac{\widehat{b}}{B^*_\parallel}(\nabla^*\mathcal{A}\times\nabla^*\mathcal{B})$$

And our second order Hamiltonian function is

$$H_{gy} = \frac{1}{2}mv^2_{\parallel gy} - \mu_{gy}B(\mathbf{X}_{gy}) - \varepsilon_\delta e\left\langle\psi_1\right\rangle - \varepsilon_\delta^2 e^2\left(\frac{1}{2mc^2}\left\langle|\mathbf{A}_1|^2\right\rangle - \frac{1}{2B(\mathbf{X}_{gy})}\partial_{\mu_{gy}}\left\langle\psi_1^2\right\rangle\right)$$

Full phase space Lagrangian constructed with addition of fields

$$\mathcal{L}^p_{gy} = \left(\frac{e}{\varepsilon c}\mathbf{A}_1 + m v_{\|gy}\hat{b}(\mathbf{X}_{gy})\right)\cdot \dot{X}_{gy} + \varepsilon\frac{mc}{e}\mu_{gy}\dot{\theta}_{gy} -$$

$$\frac{1}{2}m v^2_{\|gy} - \mu_{gy}B(\mathbf{X}_{gy}) - \varepsilon_\delta e\langle\psi_1\rangle - \varepsilon^2_\delta e^2\left(\frac{1}{2mc^2}\langle|\mathbf{A}_1|^2\rangle - \frac{1}{2B(\mathbf{X}_{gy})}\partial_{\mu_{gy}}\left(\langle\psi_1^2\rangle - \langle\psi_1\rangle^2\right)\right)$$

$$\mathcal{L}^p_{fk} = \left(m_0\mathbf{v}_0 + \frac{e}{c}A(x_0,t)\right)\cdot\dot{x}_0 - \frac{1}{2}m_0|\mathbf{v}_0|^2 + e\phi(x_0,t)$$

$$\mathcal{L}^f = \frac{1}{8\pi}\int_v d^3x_0\left(\varepsilon^2_\delta|\nabla\phi_1(x_0)|^2 - |\nabla\times[\mathbf{A}_0(x_0) + \varepsilon_\delta\mathbf{A}_1(x_0,t)]|^2 + \varepsilon_\delta\frac{2}{c}\lambda(x_0,t)\nabla\cdot\mathbf{A}_1(x_0,t)\right)$$

The action becomes

$$\mathcal{S} = \int F(X_{gy}, v_{\|gy}, \mu_{gy}, t)\mathcal{L}^p_{gy}d\Omega dt + \int F(x_0, v_0, t)\mathcal{L}^p_{fk}d\Omega' dt + \int \mathcal{L}^f dt$$

Variational principle

Characteristics are derived as variations with respect to gyrokinetic phase space

$$\dot{\mathbf{X}}_{gy} = \{\mathbf{X}_{gy}, H_{gy}\} = \frac{B^*}{mB^*_\parallel}\frac{\partial H_{gy}}{\partial v_{\parallel gy}} + \frac{c\hat{b}}{eB^*_\parallel}\varepsilon\nabla^* H_{gy}$$

$$\dot{v}_{\parallel gy} = \left\{v_{\parallel gy}, H_{gy}\right\} = \frac{B^*}{mB^*_\parallel}\cdot\left(\nabla v_{\parallel gy}\frac{\partial H_{gy}}{\partial v_{\parallel gy}} - \nabla^* H_{gy}\right) - \frac{c\hat{b}}{eB^*_\parallel}\varepsilon\left(\nabla^* v_{\parallel gy} \times \nabla^* H_{gy}\right)$$

$$\dot{\mu}_{gy} = \{\mu_{gy}, H_{gy}\} = -\frac{e}{mc}\frac{1}{\varepsilon}\partial_{\theta_{gy}} H_{gy} + \frac{B^*}{mB^*_\parallel}\nabla\mu_{gy} - \frac{c\hat{b}}{eB^*_\parallel}\varepsilon\left(\nabla\mu_{gy} \times \nabla^* H_{gy}\right)$$

$$= -\frac{e}{mc}\frac{1}{\varepsilon}\partial_{\theta_{gy}} H_{gy} = 0$$

$$\dot{\theta}_{gy} = \frac{e}{mc}\frac{1}{\varepsilon}\partial_{\mu_{gy}} H_{gy} + \frac{\nabla^*\theta_{gy}}{\nabla^* X_{gy}} \cdot \frac{\partial \mathbf{X}_{gy}}{\partial t}$$

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Conservation equation

We can also write down the gyrokinetic Vlasov equation in the conservation form. Taking in consideration the Jacobian of the gyrokinetic coordinate transformation $\mathcal{J}(\Omega) = B_{\parallel}^{*}(X_{gy}, v_{\parallel gy}, \mu_{gy}, \theta_{gy})/m$, where Ω represents the total gyrokinetic phase space. The gyrocenter phase space conservation law can be derived as following

$$\frac{\partial}{\partial \Omega} \cdot [\mathcal{J}(\Omega)\{\Omega, H_{gy}(\Omega, t)\}] = 0$$

Considering we have

$$\mathcal{J}(\Omega)F(\Omega, t) = \int d^6\Omega_0 \mathcal{J}(\Omega_0)F(\Omega_0, t_0)\delta^6[\Omega - \Omega(\Omega_0, t_0; t)]$$

And

$$\frac{d\Omega}{dt} = \{\Omega, H_{gy}(\Omega, t)\}$$

We have finally

$$\frac{\partial}{\partial t}[\mathcal{J}(\Omega)F(\Omega, t)] + \frac{\partial}{\partial \Omega} \cdot [\mathcal{J}(\Omega)F(\Omega, t)\{\Omega, H_{gy}(\Omega, t)\}] = 0$$

$$\left[\frac{\partial}{\partial t} + \{\Omega, H_{gy}(\Omega, t)\}\frac{\partial}{\partial \Omega}\right] F(\Omega, t) = 0$$

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Coulomb Gauge

The gauge fixing is a mathematical maneuver used to remove redundant degrees of freedom in any field theory. In our case we have

$$\frac{\delta}{\delta\lambda}S = \frac{\delta}{\delta\lambda}\int\left\{\frac{1}{8\pi}\int_v d^3x_0\varepsilon_\delta\frac{2}{c}\lambda(x_0,t)\nabla\mathbf{A}_1(x_0,t)\right\}dt = 0$$

$$\frac{\delta}{\delta\lambda}\left\{\lambda(x_0,t)\nabla\cdot\mathbf{A}_1(x_0,t)\right\} = 0$$

$$\frac{\delta}{\delta\lambda}\lambda(x_0,t)\nabla\cdot\mathbf{A}_1(x_0,t) + \lambda(x_0,t)\nabla\cdot\left(\frac{\delta}{\delta\lambda}\mathbf{A}_1(x_0,t)\right) = 0$$

$$\nabla\cdot\mathbf{A}_1(x_0,t) = 0$$

which is the Columb gauge condition.

Variation with respect to fields

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- Electric potential → Poisson equation

$$\nabla_\perp^2 \phi_1 - \nabla_\perp \cdot \left(\frac{1}{8\pi} \frac{\rho_{th}^2}{\lambda_D} \nabla_\perp \phi_1 - 4 v_\parallel \frac{mc}{eB} \eta_0 \nabla_\perp A_{1\parallel} - \frac{6}{Be^2} \sqrt{\frac{2\mu_{gy} mc^2}{B}} \eta_0 \nabla_\perp A_{1\perp} \right) = \langle \eta \rangle$$

  - Electrostatic Poisson equation recovered in the absence of perturbed magnetic potential

- Magnetic potential → Split Ampere equation (good for high β plasma)

$$A_1 \to A_{1\parallel} + A_{1\perp}$$

$$\frac{\delta}{\delta A_{1\parallel}} \mathcal{L} \circ \hat{A}_{1\parallel} = \frac{\delta}{\delta A_{1\parallel}} \int \left\{ -\varepsilon_\delta e \langle \psi_1 \rangle - \varepsilon_\delta^2 e^2 \left(\frac{1}{2mc^2} \left\langle |\mathbf{A}_1|^2 \right\rangle - \frac{1}{2B(\mathbf{X}_{gy})} \partial_{\mu_{gy}} \langle \psi_1^2 \rangle \right) \right. \tag{85}$$

$$\left. - \frac{1}{8\pi} \left| \nabla \times [\mathbf{A}_0(x) + \varepsilon_\delta \mathbf{A}_1(x,t)] \right|^2 + \varepsilon_\delta \frac{2}{c} \lambda(x,t) \nabla \cdot \mathbf{A}_1(x,t) \right\} d\Omega$$

Variation with respect to fields

The parallel component of the Ampere law becomes

$$\frac{\varepsilon_\delta}{4\pi}\int dV \nabla\times\hat{A}_{1\|}\cdot B - \frac{\varepsilon_\delta^2}{4\pi}\int dV \nabla\times\hat{A}_{1\|}\cdot\nabla\times A_{1\|} +$$
$$\varepsilon_\delta e\int d\Omega F\frac{v_{\|gy}}{c}\left\langle\hat{A}_{1\|}\right\rangle - \varepsilon_\delta^2\frac{e^2}{mc^2}\int d\Omega F\left\langle A_{\|1}\hat{A}_{\|1}\right\rangle -$$
$$\varepsilon_\delta^2\int d\Omega F\frac{v_{\|gy}}{c}\frac{e^2}{B}\partial_{\mu_{gy}}\left(\left\langle\psi_1\hat{A}_{1\|}\right\rangle - \left\langle\psi_1\right\rangle\left\langle\hat{A}_{1\|}\right\rangle\right) = 0$$

$$\nabla\times\hat{A}_{1\|}\left(\mu_{gy} - \frac{\varepsilon_\delta}{4\pi}\int dV\left(B - \varepsilon_\delta\nabla A_{1\|}\right)\right) +$$
$$\hat{A}_{1\|}F\left(\varepsilon_\delta e\frac{\bar{v}_\|}{c} - \frac{\varepsilon_\delta^2}{mc^2}e^2\left\langle A_{1\|}\right\rangle - \partial_\mu\left(\frac{\varepsilon_\delta^2}{B}e^2\left(\frac{\phi}{2}\frac{\bar{v}_\|}{c} - \frac{\bar{v}_\|^2}{c^2}A_{1\|}\right)\right)\right) = 0$$

Variation with respect to fields

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And the perpendicular component of the Ampere law is

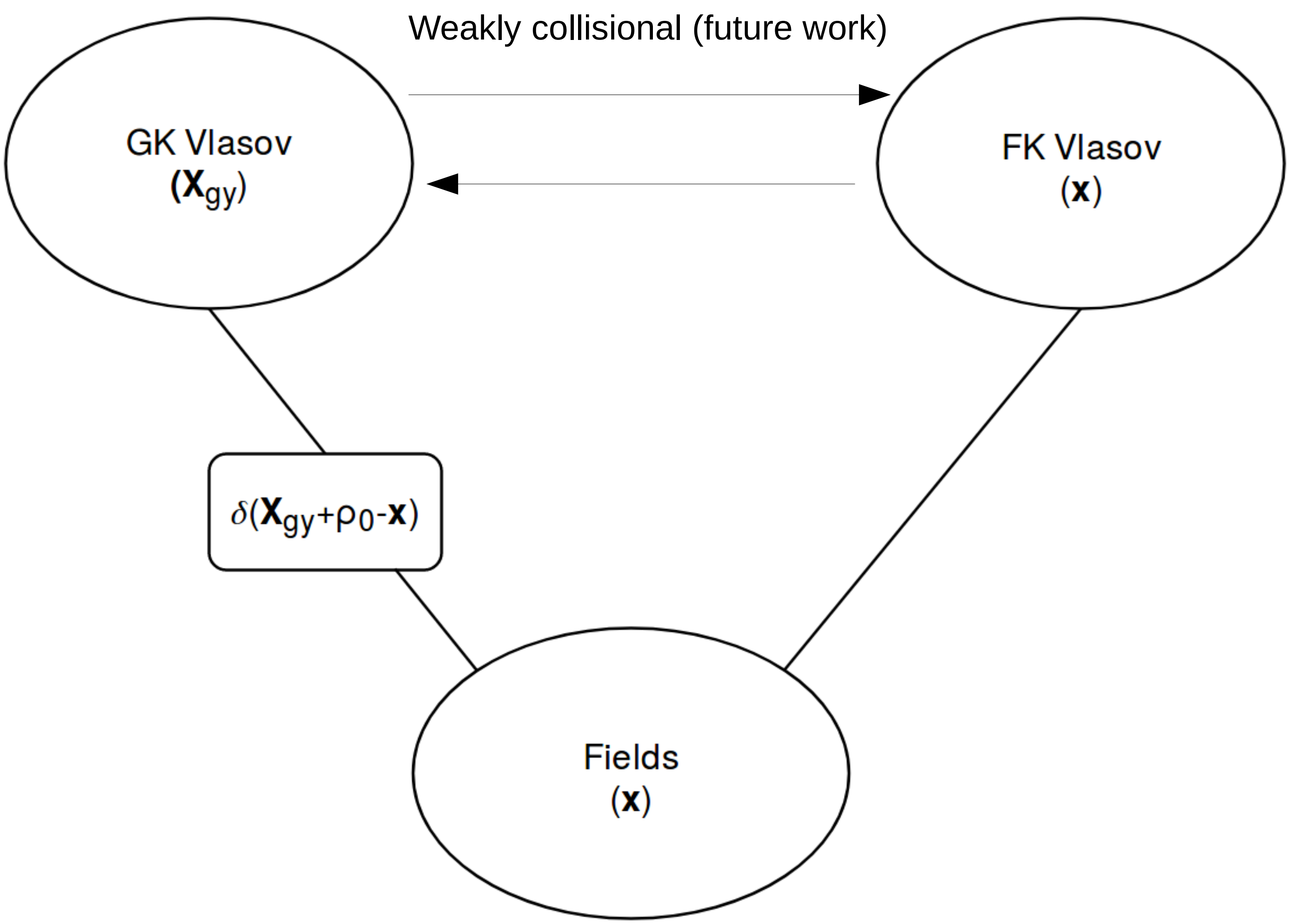
$$\frac{\varepsilon_\delta^2}{4\pi}\int dV(\nabla\times A_1)(\nabla\times\hat{A}_1)+\varepsilon_\delta e\int d\Omega F\sqrt{\frac{2\mu_{gy}B}{mc^2}} < \hat{\perp}\cdot\hat{A}_{1\perp} >$$
$$-\frac{\varepsilon_\delta m e^2}{mc^2} < A_1\cdot\hat{A}_{1\perp} > -\varepsilon_\delta^2\int d\Omega F\frac{e}{B}\partial_{\mu_{gy}}\left(\sqrt{\frac{2\mu_{gy}B}{mc^2}}\left(< \psi_1\hat{\perp}\hat{A}_{1\perp} > - < \psi_1 >< \hat{\perp}\cdot\hat{A}_{\perp 1} >\right)\right)+$$
$$\mathcal{O}(\varepsilon_\delta^3)=0$$

or

$$\nabla^2 A-\varepsilon_\delta\nabla^2 A_1=\varepsilon_\delta\int d\Omega e F\left\{\frac{1}{\varepsilon_\delta}\sqrt{\frac{2\mu_{gy}B}{mc^2}} < \hat{A}_{1\perp} > -\frac{e^2}{mc^2} < A_1\cdot\hat{A}_{1\perp} > -\frac{1}{B}\partial_{\mu_{gy}}\sqrt{\frac{2\mu_{gy}B}{mc^2}} < \psi_1\cdot\hat{A}_{1\perp} >\right\}$$

$$=j_{gk}(X_{gy},v_{\parallel gy},\mu_{gy},\theta)$$

Ions dynamics is connected to electron through field equations



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Summary

- Consistent theoretical framework to describe kinetic physics of Space and Astrophysical plasma, as well as tokamak edge.
- Geometric derivation ensures symplectic structure preservation → Symplectomorphism.
- Gyrokinetic electrons reduce computing time whilst preserving kinetic effects.
- Fully kinetic ions interact with gyrokinetic electrons through electromagnetic field equations.
- Next step: Numerical implementation!

Thank you