Regularity and anomalous entropy dissipation in a Vlasovian plasma

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Energy dissipation for the incompressible Euler equations 1/4

If $u \in L^3([0,T] \times \mathbb{T}^3)$ is a weak solution of the incompressible Euler equations on the 3-torus,

$$\partial_t u + u \cdot \nabla u = -\nabla p, \quad \nabla \cdot u = 0, \quad on\ \mathbb{T}^3,$$

then it was proved (Duchon–Robert-03) that the local energy balance holds in $\mathcal{D}'([0,T] \times \mathbb{T}^3)$ (space of distributions):

$$\partial_t \left(\frac{1}{2}|u|^2 \right) + \nabla \cdot \left(u \left[\frac{1}{2}|u|^2 + p \right] \right) = -D(u).$$

$D(u)$ is a defect distribution which for classical (smooth) solutions vanishes, implying local (and thus global) energy conservation. This defect term is given by

$$D(u)(t,x) = \lim_{\varepsilon \to 0} \frac{1}{4} \int_{\mathbb{T}^3} dr\, \nabla \varphi^\varepsilon(r) \cdot \delta u(r) |\delta u(r)|^2 \quad in\ \mathcal{D}'([0,T] \times \mathbb{T}^3), \quad where, \tag{1}$$

- $\delta u(r) = \delta u(t,x;r) = u(t,x+r) - u(t,x)$ is the velocity increment,
- $\varphi^\varepsilon(r) = \varepsilon^{-3} \varphi(x/\varepsilon)$, with $\varphi \in \mathscr{C}_0^\infty$, even, non-negative and with unit integral.

This relation is closely connected to an exact result in fluid turbulence theory, called the "Karman–Howarth–Monin relation":

$$\nabla \cdot \langle \delta u(r) |\delta u(r)|^2 \rangle = -4\bar{\epsilon}, \quad where, \tag{2}$$

- $\langle \cdot \rangle$ is a statistical (ensemble) average,
- $\bar{\epsilon} = \nu \langle |\nabla u^\nu| \rangle$ is the mean energy dissipation for a Navier-Stokes solution u^ν. $\bar{\epsilon} = \nu \langle |\nabla u^\nu| \rangle$ is assumed to remain finite and not null as the viscosity ν tend to zero.

Energy dissipation for the incompressible Euler equations 2/4

We observe that (1) corresponds to a weak formulation of (2) except that :

- (1) corresponds to the space average of an indivual realization.
- (2) corresponds to an ensemble average.
- Then (1) is stronger than (2) because there is no ensemble average.
- (2) is essentially equivalent to (1) if $\langle D \rangle = \bar{\epsilon}$.

Using the Karman–Howarth–Monin relation, Kolmogorov proved the “4/5-law” for longitudinal velocity increments, which is one of the most important result in fully developped turbulence because it is exact and nontrivial:

$$\langle |\delta u_L(r)|^3 \rangle = -\frac{4}{5}\bar{\epsilon}|r|.$$

Here $\delta u_L(t, x; r)$ is the longitudinal velocity increment defined by,

$$\delta u_L(t, x; r) = \hat{r} \cdot \delta u(t, x; r)$$

with $\hat{r}$ the unit longitudinal vector.

This relation was proved from Navier–Stokes eqs., under conditions of statistical homogeneity and local isotropy, and with the assumption that energy dissipation $\bar{\epsilon}$ remains finite and non null in the zero-viscosity limit. This type of relation can also be obtained from dimensional analysis.

Energy dissipation for the incompressible Euler equations 3/4

Let $|\cdot|_{\mathscr{C}^{0,\alpha}}$ be the semi-norm of the Hölder space $\mathscr{C}^{0,\alpha}$, defined by

$$|u|_{\mathscr{C}^{0,\alpha}} := \sup_{x \neq y} \frac{|u(x) - u(y)|}{|x - y|^{\alpha}}, \quad \alpha \in (0,1).$$

Then,

$$u \in \mathscr{C}^{0,\alpha} \text{ if and only if } \; |u|_{\mathscr{C}^{0,\alpha}} < \infty.$$

If

$$u \in \mathscr{C}^{0,\alpha}([0,T] \times \mathbb{T}^3),$$

from the relation,

$$D(u)(t,x) = \lim_{\varepsilon \to 0} \frac{1}{4} \int_{\mathbb{T}^3} dr\, \nabla \varphi^{\varepsilon}(r) \cdot \delta u(r) |\delta u(r)|^2,$$

we obtain,

$$D(u)(t,x) = \mathcal{O}(\varepsilon^{3\alpha - 1}), \quad \textit{and} \quad D(u) \to 0,\ \textit{as}\ \varepsilon \to 0, \quad \textit{if}\ \alpha > 1/3.$$

Onsager (1949) gives a semi-formal proof of the conjecture which by now carries its name:

Any weak solution which belongs to the space $\mathscr{C}^{0,\alpha}$ with $\alpha > 1/3$ conserves the energy.

This critical regularity exponent $1/3$, called the Onsager exponent could be already read from the Kolmogorov $4/5$-law.

Energy dissipation for the incompressible Euler equations 4/4

- Complete mathematical proofs were given in Constantin–E–Titi in (1994) (see also Eyink-94).

- Several other extensions in various directions:

  1. Optimized Besov spaces:
     Duchon–Robert-03, Cheskidov–Constantin–Friedlander–Shvydkoy-08, Fjordholm–Wiedemann-18, ...

  2. Other fluid models as compressible Euler, Navier–Stokes, MHD, hyperbolic systems of conservation laws:
     Cafilsh–Klapper–Steele-97, Leslie–Shvydkoy-16, Feireisl–Gwiazda–Gwiazda–Swierczewska–Wiedmann-17, Yu-17, Bardos–Gwiazda–Gwiazda–Swierczewska–Titi–Wiedmann-18, Drivas–Eyink-18, Gwiazda–Michálek–Gwiazda–Swierczewska-18, ...

  3. Boundary domains & vanishing viscosity limit:
     Robinson–Rodrigo–Skipper-17-18, Bardos–Titi-18, Bardos–Titi–Wiedemann-18, Drivas–Nguyen-18, ...

- A series of recent papers in 2015–2018 of Buckmaster, Isett, De Lellis, Szkelyhidi and Vicol have shown, for $\alpha < 1/3$, the existence (by a constructive proof) of non-unique wild solutions of incompressible Euler equations in $\mathscr{C}^{0,\alpha}([0, T] \times \mathbb{T}^3)$, which dissipate kinetic energy.

1. Energy conservation vs. anomalous dissipation in the incompressible Euler equations: a very short review

2. **Entropies conservation vs. anomalous dissipation in the relativistic Vlasov-Maxwell system**

The relativistic Vlasov–Maxwell system

The dimensionless relativistic Vlasov–Maxwell system reads,

$$\textit{Vlasov}: \quad \partial_t f + v\cdot\nabla_x f + (E + v\times B)\cdot\nabla_\xi f = 0, \tag{3}$$

$$\textit{Maxwell}: \quad \partial_t E - \nabla\times B = -j, \quad \partial_t B + \nabla\times E = 0, \quad \nabla\cdot E = \rho, \quad \nabla\cdot B = 0, \tag{4}$$

- $t\in\mathbb{R}$, $x\in\mathbb{R}^3$, $\xi\in\mathbb{R}^3$, and $v = \xi/\sqrt{1+|\xi|^2}$.
- $f = f(t,x,\xi)$ satisfies the Vlasov equation (3) with the Lorentz force $F_L = E + v\times B$.
- The electromagnetic field $E = E(t,x)$ and $B = B(t,x)$ satisfies Maxwell's equations (4).
- Coupling between (3) and (4) through the source terms of (4), i.e. charge density $\rho = \rho(t,x)$ and the current density $j = j(t,x)$, defined by:

$$\rho(t,x) = \int_{\mathbb{R}^3} f(t,x,\xi)\,d\xi, \quad j(t,x) = \int_{\mathbb{R}^3} v(\xi) f(t,x,\xi)\,d\xi. \tag{5}$$

- The initial value problem associated to the system (3)-(5)

$$f(0,x,\xi) = f_0(x,\xi) \geq 0,$$

$$E(0,x) = E_0(x), \;\; B(0,x) = B_0(x), \;\; \nabla\cdot E_0 = \rho_0 = \int_{\mathbb{R}^3} f_0\,d\xi, \;\; \nabla\cdot B_0 = 0.$$

Basic formal properties for smooth solutions

The relativistic Vlasov–Maxwell system (3)-(4) satisfies some formal conservation laws, summarized in

Proposition

Let (f, E, B) *be a smooth solution, vanishing at infinity, to the relativistic Vlasov–Maxwell system* (3)-(4)*. Then the following a priori estimates hold:*

1. *(Maximum principle).* $0 \le m \le f_0 \le M < \infty$ *implies* $m \le f(t) \le M$ *for all* $t > 0$.
2. *(L^p-norm conservation). For all* $t \ge 0$*, and* $1 \le p \le \infty$*, one has,* $\|f(t)\|_{L^p(\mathbb{R}^6)} = \|f_0\|_{L^p(\mathbb{R}^6)}$.
3. *(Entropies). For any function* $\mathcal{H} \in \mathscr{C}^1(\mathbb{R}^+; \mathbb{R}^+)$*, one has for all* $t \ge 0$,
$$\frac{d}{dt} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \mathcal{H}(f(t))\, d\xi dx = 0.$$
4. *(Energy conservation). For all* $t \ge 0$ *one has,*
$$\frac{d}{dt} \left(\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} (\gamma(\xi) - 1) f(t)\, d\xi dx + \frac{1}{2} \int_{\mathbb{R}^3} (|E(t)|^2 + |B(t)|^2)\, dx \right) = 0.$$
5. *(Momentum conservation). For all* $t \ge 0$ *one has,*
$$\frac{d}{dt} \left(\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \xi f(t)\, d\xi dx + \int_{\mathbb{R}^3} (E(t) \times B(t))\, dx \right) = 0.$$

The renormalization property 1/2

The Vlasov equation (3) has, at least formally, infinitely many invariants. Indeed, let $\mathcal{H} : \mathbb{R} \to \mathbb{R}$ be any smooth function. Multiplying (3) with $\mathcal{H}'(f)$ and applying the chain rule, we then obtain,

$$\partial_t \mathcal{H}(f) + v \cdot \nabla_x \mathcal{H}(f) + (E + v \times B) \cdot \nabla_\xi \mathcal{H}(f) = 0. \tag{6}$$

- A solution f to (3) in the sense of distributions is said to be a renormalized solution if for any smooth nonlinear function $\mathcal{H}$, f also solves (6) in the sense of distributions.
- For smooth solutions (f, E, B), renormalization property or entropies conservation are simply the consequence of the chain rule.
- For weak (not regular) solutions, the use of the chain rule is not always justified. Then arises the question about the minimal regularity needed for weak solutions to guarantee such properties.
- The renormalization property appeared as a fondamental tool, among other fields, in
  1. the well-posedness (existence and uniqueness) of passive advection equations and ODEs with rough vector fields (DiPerna–Lions-89, Ambrosio-04). From Cauchy–Lipschitz–Picard theorem, the passive transport equation,
     $$\partial_t u + b \cdot \nabla u = 0, \quad u : [0, T] \times \mathbb{R}^d \to \mathbb{R}, \; b : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d, \; \text{and}$$
     $$\text{the associated ODE}, \; \partial_t X(t, x) = b(t, X(t, x)), \; t \in [0, T], \; X(0, x) = x \in \mathbb{R}^d,$$
     are well-posed if b is a Lipschitz (in space) vector field, i.e. $b \in L^1_t W^{1,\infty}_x$.
     DiPerna–Lions-89 (resp. Ambrosio-04) show that it is still true when $b \in L^1_t W^{1,p}_x$ with $p \geq 1$, (resp. $b \in L^1_t BV_x$)
  2. the theory of weak solutions of collisionless kinetic equations such as the Vlasov–Poisson system, where initial data have very poor integrability assumptions (DiPerna–Lions-88, Bouchut-01).

The renormalization property 2/2

- For bounded distribution function f, renormalization property holds for Vlasov–Poisson because elliptic regularity of the Poisson equation leads to $E \in W^{1,p}$, with $p > 1$ (DiPerna–Lions-88).
- For the Vlasov–Maxwell system the only available global existence result is in DiPerna–Lions-89, where the authors have constructed weak solutions such that

$$f \in L_t^\infty L_{x,\xi}^1 \cap L_{x,\xi}^\infty, \qquad E, B \in L_t^\infty L_x^2,$$

  and for which it is not possible to show the renormalization property.
- The best electromagnetic-field regularity, obtained so far for the DiPerna–Lions weak solutions, is in Besse–Bechouche-18, where the authors show that the electromagnetic field (E, B) belongs to $H^s_{\mathrm{loc}}(\mathbb{R}_*^+ \times \mathbb{R}^3)$, with $s = 6/(13 + \sqrt{142})$, if the macroscopic kinetic energy is in L^2.
- Here, we show that the renormalization property holds for an electromagnetic field with only a fractional derivative in some Lebesgue spaces, i.e. $E, B \in L_t^\infty W_x^{\beta,q}$, with $0 < \beta < 1$ and $1 \leq q \leq \infty$.
  To compensate this loss of derivative for the electromagnetic field, the density f requires additional smoothness, typically fractional Sobolev differentiability in phase-space, i.e. $f \in L_t^1 W_{x,\xi}^{\alpha,p}$, with $0 < \alpha < 1$ and $1 \leq p \leq \infty$.
- We determine Onsager type exponents α and β, which ensure conservation of all entropies and guarantee that the renormalization property holds.
- Moreover we show that total energy is preserved, if the macroscopic kinetic energy is in L^2.

Some notation and definitions

- Lebesgue spaces: for $1 \le p \le \infty$, the function f is in the Lebesgue spaces $L^p(\mathbb{R}^n)$ iff

$$\|f\|_{L^p(\mathbb{R}^n)} := \left(\int_{\mathbb{R}^n} |f(x)|^p dx\right)^{1/p} < +\infty,\ 1 \le p < \infty,\ \text{ and } \|f\|_{L^\infty(\mathbb{R}^n)} := \sup_{x \in \mathbb{R}^n} |f(x)|,\ p = \infty.$$

- Fractional-order Sobolev spaces: for $\alpha \in (0,1)$ and $1 \le p \le \infty$ the function f is in the fractional-order Sobolev space $W^{\alpha,p}(\mathbb{R}^n)$ iff

$$\|f\|_{W^{\alpha,p}(\mathbb{R}^n)} := \|f\|_{L^p(\mathbb{R}^n)} + \left(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|f(x)-f(y)|^p}{|x-y|^{n+\alpha p}} dx\right)^{1/p} < +\infty,\quad 1 \le p < \infty, \quad \text{and}$$

$$\|f\|_{W^{\alpha,\infty}(\mathbb{R}^n)} := \sup\left\{\|f\|_{L^\infty(\mathbb{R}^n)},\quad \sup_{x \neq y \in \mathbb{R}^n} \frac{|f(x)-f(y)|}{|x-y|^\alpha}\right\} < +\infty,\quad p = \infty.$$

- $H^\alpha := W^{\alpha,2}$
- We define the functional space L^1_γ such that,

$$L^1_\gamma = \left\{ f \ge 0 \ \text{ a.e } \ \middle| \ \ \|f\|_{L^1_\gamma(\mathbb{R}^6)} := \int_{\mathbb{R}^6} \gamma f \, dx d\xi < +\infty \right\}.$$

- We define the function space $\mathscr{E}$ such that,

$$\mathscr{E} = \left\{ \mathcal{H} : \mathbb{R}^+ \mapsto \mathbb{R}^+;\ \mathcal{H} \text{ is non-decreasing},\ \ \mathcal{H} \in \mathscr{C}^1(\mathbb{R}^+;\mathbb{R}^+),\ \ \lim_{\sigma \to +\infty} \frac{\mathcal{H}(\sigma)}{\sigma} = +\infty \right\}.$$

Global-in-time existence of weak solutions

Theorem (DiPerna–Lions-89)

Let $f_0 \in L^1_\gamma \cap L^\infty(\mathbb{R}^6)$, and $E_0,\ B_0 \in L^2(\mathbb{R}^3)$, be initial conditions which satisfy the constraints, $\nabla \cdot B_0 = 0,\ \nabla \cdot E_0 = \rho_0 = \int_{\mathbb{R}^3} f_0\, d\xi,$ in $\mathcal{D}'(\mathbb{R}^3)$. Then, there exists a global-in-time weak solution of the relativistic Vlasov–Maxwell system, i.e. there are functions,

$$f \in L^\infty\left(\mathbb{R}^+_t; L^1_\gamma \cap L^\infty(\mathbb{R}^6_{x,\xi})\right), \quad E,\, B \in L^\infty\left(\mathbb{R}^+_t; L^2(\mathbb{R}^3_x)\right), \quad \rho,\, j \in L^\infty\left(\mathbb{R}^+_t; L^{4/3}(\mathbb{R}^3_x)\right),$$

such that (f, E, B) satisfy (3)-(4) in the sense of distributions. Futhermore, the total mass,

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} f(t) dx d\xi, \ \text{ is independent of time, and one has,}$$

$$\|f(t)\|_{L^p(\mathbb{R}^6)} \le \|f_0\|_{L^p(\mathbb{R}^6)} \ \text{ a.e. } t \ge 0, \ \text{ for } 1 \le p \le +\infty, \quad \text{and} \quad \mathcal{E}(t) \le \mathcal{E}_0 < \infty \ \text{ a.e. } t \ge 0,$$

$$\text{with the definition, } \mathcal{E}(t) := \int_{\mathbb{R}^6} \gamma f(t)\, d\xi dx + \frac{1}{2}\int_{\mathbb{R}^3} (|E(t)|^2 + |B(t)|^2)\, dx.$$

Remark

Using lower semi-continuity, weak solutions of DiPerna–Lions-89 satisfy, for all $\mathcal{H} \in \mathscr{C}^1(\mathbb{R}^+; \mathbb{R}^+)$,

$$\int_{\mathbb{R}^6} \mathcal{H}(f(t))\, d\xi dx \le \int_{\mathbb{R}^6} \mathcal{H}(f_0)\, d\xi dx, \quad \text{for } t \ge 0.$$

Theorem (Bardos–Besse–Nguyen-19)

Let (f, E, B) be a weak solution of the relativistic Vlasov–Maxwell system (3)-(4), given by the DiPerna–Lions-89 theorem. Assume that with,

$$\alpha, \beta \in \mathbb{R}, \qquad 0 < \alpha,\ \beta < 1, \qquad \textit{and} \qquad \alpha\beta + \beta + 3\alpha - 1 > 0, \tag{7}$$

this weak solution satisfies for some $(p, q) \in \mathbb{N}_*^2$ with,

$$\begin{aligned} &\frac{1}{p} + \frac{1}{q} = \frac{1}{r} \leq 1 \quad \textit{if} \quad 1 \leq p, q < \infty, \\ &1 \leq r < \infty \textit{ is arbitrary if } p = q = \infty, \end{aligned} \tag{8}$$

the supplementary regularity hypotheses,

$$f \in L^\infty\Big(0, T; W^{\alpha,p}(\mathbb{R}^6)\Big), \qquad \textit{and} \qquad E, B \in L^\infty\Big(0, T; W^{\beta,q}(\mathbb{R}^3)\Big). \tag{9}$$

Then for any entropy function $\mathcal{H} \in \mathscr{C}^1(\mathbb{R}^+; \mathbb{R}^+)$, we have the renormalization property,

$$\partial_t(\mathcal{H}(f)) + \nabla_x \cdot (v\mathcal{H}(f)) + \nabla_\xi \cdot (F_L \mathcal{H}(f)) = 0, \quad \textit{in } \mathcal{D}'((0, T) \times \mathbb{R}^6). \tag{10}$$

Renormalization property and entropies conservation 2/3

Theorem (Bardos–Besse–Nguyen-19)

Moreover, if $\mathcal{H} \in \mathscr{E}$ and the map, $t \mapsto f(t,\cdot,\cdot)$ is uniformly integrable in $\mathbb{R}^6$, for a.e. $t \in [0;T]$, then we have the local entropy conservation laws,

$$\partial_t \left(\int_{\mathbb{R}^3} d\xi \, \mathcal{H}(f) \right) + \nabla_x \cdot \left(\int_{\mathbb{R}^3} d\xi \, v\mathcal{H}(f) \right) = 0, \quad \text{in } \mathcal{D}'((0,T) \times \mathbb{R}^3), \tag{11}$$

$$\partial_t \left(\int_{\mathbb{R}^3} dx \, \mathcal{H}(f) \right) + \nabla_\xi \cdot \left(\int_{\mathbb{R}^3} dx \, F_L \mathcal{H}(f) \right) = 0, \quad \text{in } \mathcal{D}'((0,T) \times \mathbb{R}^3), \tag{12}$$

and the global entropy conservation law,

$$\int_{\mathbb{R}^6} \mathcal{H}(f(t,x,\xi)) \, d\xi dx = \int_{\mathbb{R}^6} \mathcal{H}(f(s,x,\xi)) \, d\xi dx, \quad \text{for } 0 < s \le t < T. \tag{13}$$

Remark

- *In fact, thi theorem is also true for the Vlasov–Poisson and the non-relativistic Vlasov–Maxwell systems, under the same regularity assumptions.*
- *Our result is almost in agreement with the structure-function scaling exponents derived in the study of dissipative anomalies in nearly collisionless plasma turbulence by Eyink-18. Our result is more general.*
- *In the non-self-consistent case, i.e. when the Lorentz force F_L is a given external force, renormalization property implies straightforwardly the uniqueness of this weak solutions.*

Renormalization property and entropies conservation 3/3

Corollary (Bardos–Besse–Nguyen-19)

Let $\beta = 6/(13+\sqrt{142})$*, and* $\alpha \in \mathbb{R}$ *solution to,* $\alpha\beta + \beta + 3\alpha - 1 > 0$*, and* $0 < \alpha < 1$.

Let (f, E, B) *be a weak solution to the relativistic Vlasov–Maxwell system* (3)-(4)*, given by DiPerna–Lions-89 theorem. Assume the additional hypotheses: initial conditions* (E_0, B_0) *belong to* $H^1(\mathbb{R}^3)$*, the distribution function* f *satisfies the supplementary integrability condition,*

$$\int_{\mathbb{R}^3} \gamma f \, d\xi \in L^\infty\left(0, T; L^2(\mathbb{R}^3_x)\right), \quad \text{and the regularity assumption,} \;\; f \in L^\infty\left(0, T; H^\alpha(\mathbb{R}^6_{x,\xi})\right). \tag{14}$$

Then for any entropy function $\mathcal{H} \in \mathscr{C}^1(\mathbb{R}^+; \mathbb{R}^+)$*, renormalization property* (10) *holds. Moreover, if* $\mathcal{H} \in \mathscr{E}$ *and the map* $t \mapsto f(t, \cdot, \cdot)$ *is uniformly integrable in* $\mathbb{R}^6$ *for a.e.* $t \in \mathbb{R}^+$*, then local entropy conservation laws* (11)-(12)*, as well as, global entropy conservation law* (13) *hold.*

Proof. Using assumptions $E_0, B_0 \in H^1(\mathbb{R}^3)$ and the L^2-bound on the macroscopic kinetic-energy density, from Theorem 1.1 of Besse–Bechouche-18, we obtain that the electromagnetic field (E, B) belongs to $H^\beta_{\mathrm{loc}}(\mathbb{R}^+ \times \mathbb{R}^3)$, with $\beta = 6/(13+\sqrt{142})$. Setting $p = q = 2$ and $\beta = 6/(13+\sqrt{142})$ in the hypotheses of Theorem of Bardos–Besse-Nguen-19, we obtain the desired result. □

Remark

From Corollary, we have entropies conservation if (α, β) *are such that,*

$$\beta = \frac{6}{13+\sqrt{142}} \simeq 0.24, \quad \text{and} \;\; 1 > \alpha > \frac{1-\beta}{3+\beta} = \frac{7+\sqrt{142}}{45+3\sqrt{142}} \simeq 0.234 \;\; (< 1/3 \;\; \textit{[fluid models]}).$$

THANK YOU FOR YOUR ATTENTION