Long-Time Behavior of Nonlinear Landau Damping

Giovanni Manfredi
School of Cosmic Physics, Dublin Institute for Advanced Studies, Dublin 2, Ireland
(Received 9 June 1997)

The evolution of an initial perturbation in a collisionless, Maxwellian plasma is studied numerically. Accurate long-time simulations (up to 1600 inverse electron plasma frequencies) show that the electric field does not decay to zero, in disagreement with recent analytical results [M. B. Isichenko, Phys. Rev. Lett. **78**, 2369 (1997)]. Instead, after some initial damping, the field amplitude starts to oscillate around an approximately constant value, and the phase-space distribution develops a vortex structure which survives throughout the simulation. [S0031-9007(97)04171-9]

PACS numbers: 52.35.Mw, 52.35.Sb, 52.65.–y

In 1946, Landau discovered that small perturbations in a uniform, Maxwellian, electrostatic plasma are exponentially damped, even when no dissipative terms are present [1]. Since then, linear Landau damping has been extensively confirmed in both experiments [2] and computer simulations [3], and has become a standard topic in most plasma physics textbooks (e.g., [4,5]). Besides, it has been shown to play an important role in many applications, such as plasma heating in fusion devices [5] and laser-plasma interactions [6].

Landau's treatment of the problem is rigorous, but strictly linear, meaning that the initial perturbation is supposed to be infinitesimally small. For a finite perturbation, only approximate analytical solutions are available [7]. O'Neil's theory, for example, predicts amplitude oscillations for the electric field, which have indeed been observed in experiments [8] and simulations, including those presented in our paper. However, O'Neil's treatment ceases to be valid for large times. Early numerical results [9], on the other hand, are not accurate enough to allow us to draw conclusions on the long-time limit.

Until recently, it was generally believed [4] that nonlinear plasma waves undergo a few amplitude oscillations and eventually approach a Bernstein-Greene-Kruskal (BGK) steady state [10]. More recent simulation results [11] seem to support this conjecture, although the evidence is not conclusive. Two papers have recently challenged this belief, claiming that the wave amplitude will eventually decay to zero. Brodin [12], starting from the Vlasov-Poisson system, develops a reduced model, which is then solved numerically. Although some overall decay is indeed shown for about two trapping oscillations, these results are still inconclusive as far as the long-time limit is concerned.

In another paper, Isichenko [13] presents a general theory predicting that Landau damping will continue indefinitely, although for large times the electric field decay is algebraic ($E \propto t^{-1}$) rather than exponential. Isichenko's algebraic decay is presented as an exact asymptotic result, valid for general initial perturbations. The purpose of our paper is to provide numerical evidence to verify the validity of this result. We present the results of some accurate numerical computations for very long times, up to 1600 inverse electron plasma frequencies. Our conclusion is that Isichenko's result is not general: There exist initial conditions for which the electric field does not decay to zero (algebraically or otherwise), but rather settles to a finite value.

Our mathematical model is the one-dimensional Vlasov-Poisson system,

$$\begin{aligned} \frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} + E(x,t)\frac{\partial f}{\partial v} &= 0\,, \\ \frac{\partial E}{\partial x} = \int_{-\infty}^{\infty} f\,dv - 1\,&, \end{aligned} \tag{1}$$

where $f(x,v,t)$ is the electron distribution function and $E(x,t)$ the electric field. In Eq. (1), and in the rest of the article, time is normalized to the inverse electron plasma frequency ω_{pe}^{-1}, space is normalized to the Debye length λ_D, and velocity is normalized to the electron thermal speed $V_{Te} = \lambda_D\omega_{pe}$. Ions are taken to be motionless, and their only role is to provide a uniform, neutralizing background. Furthermore, periodic boundary conditions are assumed in x, L being the box length. Oscillations are excited by initializing a single Fourier mode, namely, the fundamental mode $k = 2\pi/L$: $f(x,v,0) = f_0(v)\,(1 + \alpha\cos kx)$, where $f_0(v) = (2\pi)^{-1/2}\exp(-v^2/2)$ is the equilibrium Maxwellian. This problem has only two dimensionless parameters, namely, the strength of the nonlinearity α and the perturbation wave number $k\lambda_D$. These can be more usefully expressed as two time scales: the Landau damping rate γ (which depends on the wave number) and the bounce time $\tau = \alpha^{-1/2}$. Linear Landau damping is valid as long as $t < \tau$; for longer times the problem is inherently nonlinear, irrespective of the initial perturbation amplitude and wave number. However, the actual long-time asymptotics need not be the same for different values of the two dimensionless parameters. It is Isichenko's conjecture that the asymptotic behavior $E \propto t^{-1}$ is universal for all values of α and $k\lambda_D$.

We now turn to the numerical study of the Vlasov-Poisson system. The simulation of nonlinear Landau damping is a difficult numerical problem since different time scales are present, and the physical effect is small and must be separated from numerical noise. It is therefore important to perform a series of checks to rule out spurious numerical artifacts. The code used for our simulation is an Eulerian code [14], which solves the Vlasov equation on a uniform mesh in phase space. It is second order accurate in space, velocity, and time. This code has been used extensively over the past two decades [3,6,11,15], and has been found to be very accurate in describing coherent phase-space structures.

We report results for a moderately nonlinear problem with parameters $\alpha = 0.05$ and $k = 0.4$, corresponding to $\gamma = 0.0661$ and $\tau = 4.47$. The real part of the frequency, from Landau's theory, is $\omega = 1.285$. The relevant numerical parameters are the time step Δt, the number of points in x and v (respectively, N and M), and the cutoff velocity $v_{\max}$ (i.e., the distribution function is set to zero for $v > |v_{\max}|$). The time step must be small enough to describe the largest frequency in the problem, which is the Landau frequency given above, therefore we take $\Delta t = 0.1$. This is a rather conservative choice: Values as high as 0.25 were used in the literature [15], still retaining good accuracy. The cutoff velocity is $v_{\max} = 6$: This is considerably larger than the phase velocity of the wave $v_{\text{phase}} = \omega/k = 3.21$. The choice of the number of points is more subtle. Since the electric field remains small, the particle motion is close to free streaming, thus developing a fine structure in velocity space. For truly free streaming particles ($E = 0$) the exact solution of the Vlasov equation is $f_k(v,t) = f_0(v)\exp(ikvt)$. If the mesh spacing in velocity space is Δv, there is a recurrence occurring at $T_R = 2\pi/(k\Delta v)$. This recurrence effect can be easily detected when simulating linear Landau damping, as it appears as a sharp spike in the electric field at T_R. Unfortunately, when nonlinearities are important, the recurrence cannot be recognized so easily, and one should only rely on the numerical results obtained up to a time much smaller than T_R. In the computations presented here, we use $M = 4000$ and $M = 8000$, yielding $T_R = 5230$ in the less favorable case, which is much larger than the total time of the run ($t = 1600$). The number of spatial points N is more difficult to estimate. Since resonant particles oscillate in the potential well, the microstructure in velocity space will generate a microstructure in x, requiring a high resolution. This is also suggested by the semianalytical results of Brodin [12]. In our simulations, we take $N = 512$ and $N = 1024$, which appears to be accurate enough for this case.

Three simulations were run with the same time step $\Delta t = 0.1$, but different meshes. Run I: $N = 512$, $M = 4000$; run II: $N = 512$, $M = 8000$; run III: $N = 1024$, $M = 4000$. Figure 1 shows the evolution of the fundamental mode of the electric field $|E_k(t)|$ for the three runs. Linear Landau damping is recovered accurately until $t \simeq 25$. The measured real part of the frequency, averaging between $t = 0$ and $t = 200$, is $\omega \simeq 1.263$, slightly smaller than the Landau frequency. After the linear stage, trapping oscillations are observed, while the maximum amplitude decreases at each oscillation. However, after $t \simeq 900$, no further decrease is observed, and the electric field goes on oscillating around an approximately constant value. The trapping oscillations are indeed predicted by O'Neil's theory [7], which applies when $\gamma\tau \ll 1$. In our case, the initial value of the latter parameter is $\gamma\tau = 0.296$; at saturation ($t \simeq 900$), the electric field is roughly $E \simeq 0.007$, which gives $\tau = 11.9$ and $\gamma\tau = 0.79$.

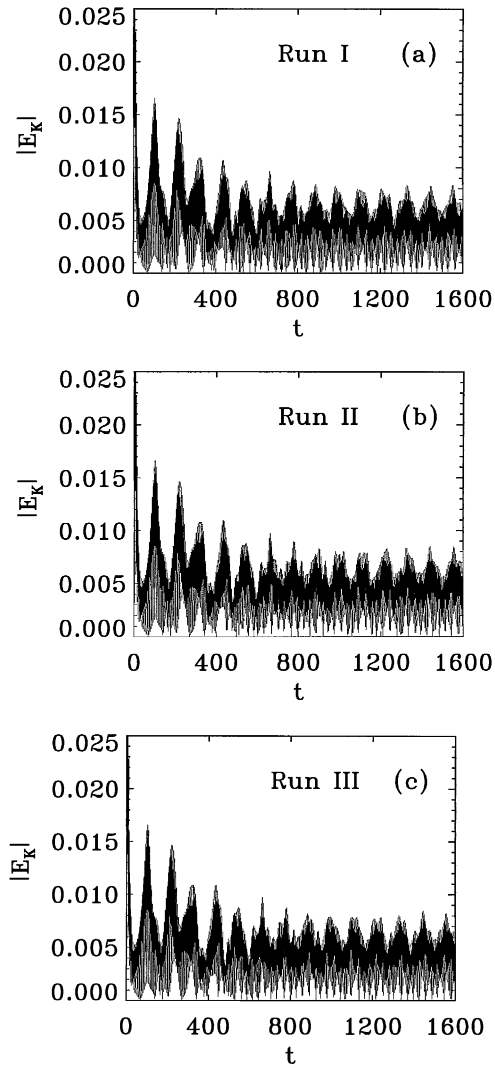

FIG. 1. Evolution of the amplitude of the electric field for the fundamental mode $k = 2\pi/L$. (a) Run I: $N = 512$, $M = 4000$; (b) run II: $N = 512$, $M = 8000$; (c) run III: $N = 1024$, $M = 4000$.

The main result of Fig. 1 is that the field does not decrease indefinitely, but finally settles to a constant value. Comparing the results obtained with different meshes shows that no qualitative difference is observed

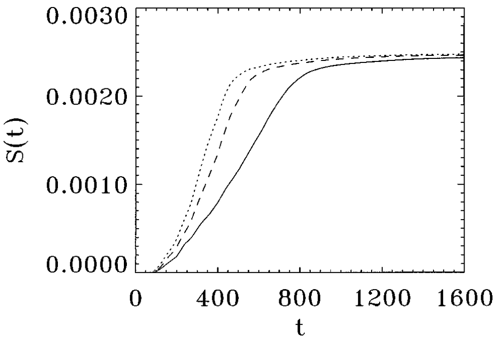

FIG. 2. Relative numerical entropy $S_{\rm rel}(t)$ for run I (dotted line); run II (solid line); run III (dashed line).

by increasing the resolution. Quantitatively, there are indeed small differences between the three runs, which is inevitable since smaller and smaller scale structures are created by the essentially free-streaming nature of the dynamics. However, there is no indication that, by increasing the resolution, a further decay would be observed. Besides, other numerical tests have been performed at even lower resolution, again showing the same qualitative behavior, provided the resolution is not too low. Reducing the time step does not change the picture either. This is in agreement with all the previous experience with the Eulerian Vlasov code used for our study [11]: Once the microstructure reaches the mesh size, it is smoothed away by numerical diffusion (essentially due to the interpolation technique used in the code), and is therefore lost. However, larger scales appear to be virtually unaffected by the small scale diffusivity. This effect can be quantified by means of a numerical "entropy," defined in the usual way $S(t) = -\int f \ln f \, dx \, dv$. Obviously, $S(t)$ is a constant for the exact Vlasov-Poisson system, Eqs. (1). However, it increases monotonically for the discrete numerical model (this is a property of the scheme, and can be proven rigorously). The evolution of the entropy is shown in Fig. 2 for the three runs {what is actually plotted is the relative entropy $S_{\rm rel} = [S(t) - S(0)]/S(0)$}. As expected, the growth is slower for the higher resolution case, although eventually the three runs saturate at the same level. This is an indication that, although the microstructure is lost more quickly for a coarser mesh, larger scales are treated with good accuracy in all three runs. Moreover, the total increase in entropy is extremely small, less than 1%. Other entropylike functionals can be used, such as $S_2(t) = -\int f^2 \, dx \, dv$, which give essentially the same result.

In phase space, the distribution function develops a vortex structure roughly at the phase velocity of the wave $v_{\rm phase} = \omega/k = 3.21$, which was already observed in previous simulations [11,15]. Such vortex structures have also been shown to arise spontaneously from a perturbed, unstable equilibrium ("bump-on-tail") [16].

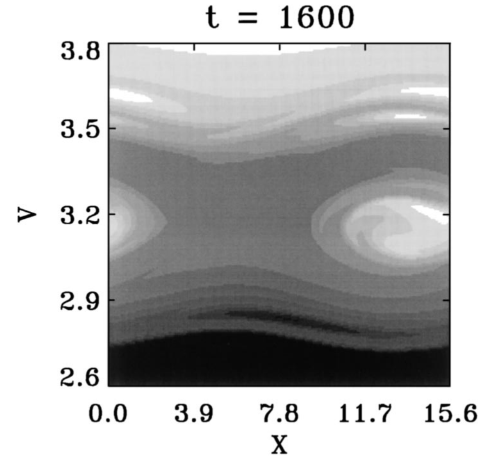

FIG. 3. Phase-space shaded plot of the distribution function in the resonant region ($v_{\rm phase} \simeq 3.21$) (run I). Darker regions correspond to regions of higher density. Regions where $f > 0.008$ are black.

The phase portrait is shown in Fig. 3—note that a similar structure can be found at the corresponding negative velocity. These structures are present up to the end of our simulation, and there is no indication that they should be eventually damped away. It appears therefore that a finite number of particles can be trapped for arbitrarily long times. The average velocity distribution strongly deviates from the initial Maxwellian in the region around $v_{\rm phase}$ (Fig. 4). However, it never settles to a plateau—rather, its slope changes periodically. This effect results in the low frequency amplitude oscillations observed in the electric field (Fig. 1) even after saturation ($t \simeq 900$).

Before comparing our simulation results with Isichenko's theory [13], it must be noted that Isichenko's proof requires the presence of at least two waves in the perturbation, while in our case only one mode was initially excited. However, since the problem is fully nonlinear, higher order modes are quickly generated by wave coupling. In the case considered above, for

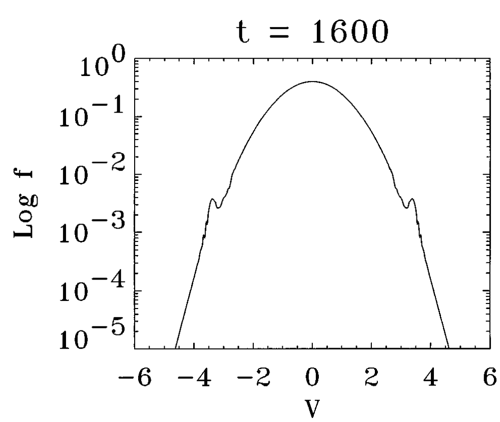

FIG. 4. Logarithmic plot of the velocity distribution averaged over x at the end of the simulation (run I).

example, the second harmonic of the electric field rapidly reaches a value of about 4×10^{-4}, i.e., roughly 20 times smaller than the fundamental mode. For an even closer comparison, we have run another case in which *two* modes are initially excited, with wave numbers $k = 0.2$ and $k = 0.4$, and the same amplitude $\alpha = 0.05$. The amplitude of the $k = 0.4$ mode (Fig. 5) behaves in a way similar to the previous case, and finally settles to a slightly smaller value with no further decay. The $k = 0.2$ mode (not shown here) is almost unaffected by linear Landau damping, and remains at an approximately constant amplitude throughout the entire run. We are therefore confident that our main result is not affected by the presence of a second wave.

The results that we have reported are partially consistent with those obtained by Brodin [12]. However, Brodin follows the evolution for only about two trapping oscillations, during which the peak of the oscillating amplitude decreases by a factor of 2. This is much too short a time to draw conclusions about the asymptotic behavior. In our simulations, the electric field decays for about five trapping oscillations, before settling to a constant amplitude.

Our results also do not contradict the stability analysis of BGK solutions performed with a similar Eulerian code [15]. In that paper, it is shown that BGK states with more than one vortex ("hole") are unstable, and evolve towards a one-vortex structure. However, BGK solutions with only one hole appear to be stable. The structure that we obtain at the end of our simulation can be viewed as a traveling BGK wave, and is equally stable.

The theoretical result of Isichenko [13], which predicts an algebraic decay for the electric field in the long-time limit ($E \propto t^{-1}$), is obviously in disagreement with our computations. In fact, Isichenko *assumes* that the electric field decays to zero, and then goes on proving that the long-time damping rate is algebraic, rather than exponential as in the linear case. An inspection of the details of the proof, however, shows that this initial assumption is crucial to the demonstration. In other words, Isichenko's proof does not exclude solutions for which the electric field remains asymptotically finite, but only proves that, if the decay continues indefinitely, it must be algebraic. Our computations show that solutions for which the field remains finite are not only possible in principle, but can actually be approached from an initial state.

It remains an open question whether the algebraic decay is ever observed as an asymptotic solution of the initial value problem. We have performed a few other simulations with wave number $k = 0.4-0.5$ and initial perturbation $\alpha = 0.1-0.25$, and we always observe a finite number of trapped particles for large times. However, all these computations are in the small $\gamma\tau$ regime. Early numerical results [9] suggested that there exists a critical $(\gamma\tau)_{\rm cr} \simeq 0.5$, such that for values larger than the critical one, the electric field amplitude is damped monotonically. These results were obtained at low resolution, and for very short times (less than $100\ \omega_{pe}^{-1}$), so that no conclusion about the long-time limit can be drawn. Simulations in this regime are also particularly delicate, since the field rapidly becomes very small. Further studies using our numerical code may help to clarify this important point.

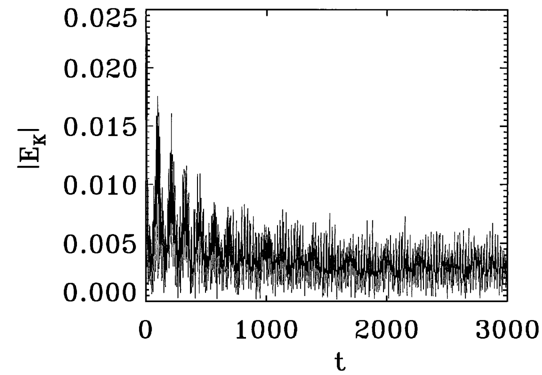

FIG. 5. Evolution of the amplitude of the electric field for the second harmonic $k = 4\pi/L = 0.4$

I am grateful to Richard Dendy for bringing some of the recent literature to my attention, and to Marc Feix and Luke Drury for several useful discussions. This work was partially supported by the Irish Fusion Association.

[1] L. D. Landau, J. Phys. (Moscow) **10**, 25 (1946).
[2] J. H. Malmberg and C. B. Wharton, Phys. Rev. Lett. **6**, 184 (1964).
[3] R. R. J. Gagné and M. M. Shoucri, J. Comput. Phys. **24**, 445 (1977).
[4] D. R. Nicholson, *Introduction to Plasma Theory* (Wiley, New York, 1983).
[5] K. Miyamoto, *Plasma Physics for Nuclear Fusion* (MIT Press, Cambridge, MA, 1987).
[6] P. Bertrand *et al.*, Phys. Rev. E **49**, 5656 (1994).
[7] T. M. O'Neil, Phys. Fluids **8**, 2255 (1965).
[8] J. H. Malmberg and C. B. Wharton, Phys. Rev. Lett. **19**, 775 (1967).
[9] J. Canosa and J. Gazdag, Phys. Fluids **17**, 2030 (1974).
[10] I. B. Bernstein, J. M. Greene, and M. D. Kruskal, Phys. Rev. **108**, 546 (1957).
[11] M. R. Feix, P. Bertrand, and A. Ghizzo, in *Advances in Kinetic Theory and Computing*, edited by B. Perthame (World Scientific, Singapore, 1994), pp. 45–81.
[12] G. Brodin, Phys. Rev. Lett. **78**, 1263 (1997).
[13] M. B. Isichenko, Phys. Rev. Lett. **78**, 2369 (1997).
[14] C. Z. Cheng and G. Knorr, J. Comput. Phys. **22**, 330 (1976).
[15] A. Ghizzo *et al.*, Phys. Fluids **31**, 72 (1988).
[16] M. Shoucri, Phys. Fluids **22**, 2038 (1979).