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1993 Eur. J. Phys. 14 101

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Quantum systems that follow classical dynamics

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Received 22 June 1992, in final form 30 December 1992

Abstract. For a special class of potentials, the dynamical evolution of any quantum wavepacket is entirely determined by the laws of classical mechanics. Here, the properties of this class are investigated both from the viewpoint of the Ehrenfest theorem (which provides the evolution of the average position and momentum), and the Wigner representation (which expresses quantum mechanics in a phase space formalism). Finally, these results are extended to the case of a charged particle in a uniform magnetic field.

Résumé Pour une classe particulière de potentiels, l'evolution de tout paquet d'onde est entièrement déterminée par les lois de la mécanique classique. On a étudié les propriétés de cette classe à l'aide du théorème d'Ehrenfest (donnant l'évolution de la position et du moment moyens) et de la représentation de Wigner (qui exprime la mécanique quantique en terme d'une distribution de probabilités dans l'espace des phases). Ces résultats sont ensuite étendus au cas d'une particule chargée dans un champs magnétique uniforme.

1. Introduction

After sixty years since the first consistent formulation of the principles of quantum mechanics (QM), a considerable amount of disagreement still exists on the very foundation of the theory. It should be noted that no other physical theory has ever undergone such a never-ending debate *about its theoretical interpretation* (apart, perhaps, from statistical mechanics: it is probably not a coincidence that a great many of the puzzling features of QM arise from its intrinsic statistical nature).

On the other hand, there is a virtually universal agreement on the practical use of the rules of QM. Indeed, the mathematical machinery of the theory *plus* the so-called Copenhagen interpretation, have proved to be two extremely good tools in the description of the microscopic world.

It is out of the scope of this article to consider the possible alternatives to the Copenhagen interpretation: readers who are interested in the philosophical implications of QM can refer to a recent review by Omnès (1992).

Here, we shall concentrate on a particular aspect of the debate, namely the conditions under which the evolution of a quantum system follows exactly the laws of classical dynamics. Moreover, we shall deal with some mathematical connections between different representations. It is our feeling that progress on the foundations of QM will also come from these 'technical' manipulations.

The paper will be structured as follows. We shall restate the well known Ehrenfest theorem, and show how, for a special class of Hamiltonians, it reduces to the Newtonian equations of motion. An analogous result will be proved to hold also for classical statistical mechanics (section 2). In section 3, we shall introduce the Wigner representation, which allows us to express QM in a phase space formalism: the previously obtained results will then be analysed from this point of view. Finally, in sections 4 and 5 the family of Hamiltonians inducing a classical behavior will be enlarged to include the case of a charged particle in a uniform magnetic field.

2. Ehrenfest's theorem

The dynamical evolution of a quantum state is given by the time-dependent Schrödinger equation:

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + V\Psi \tag{1}$$

A remarkably simple analogy between quantum and classical evolution laws comes from the Ehrenfest theorem, which is undoubtely the oldest result in this domain. It is also the most widely known, and it appears in almost all textbooks on elementary QM (Messiah 1965, Schiff 1965, with the remarkable exception of Landau and Lifschitz 1969). The theorem reads as:

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle x\rangle = \frac{\langle p\rangle}{m} \qquad \frac{\mathrm{d}}{\mathrm{d}t}\langle p\rangle = -\left\langle\frac{\partial V}{\partial x}\right\rangle \tag{2}$$

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where the symbol $\langle . \rangle$ denotes mean values as usually defined in QM. As a matter of fact, equation (2) are void of meaning in this form, since, in order to calculate the mean value of $\partial V/\partial x$, we need knowledge of the entire wavefunction.

Nevertheless, there is a special case for which the system (2) becomes self-contained, namely when the following condition is fulfilled:

$$\left\langle \frac{\partial V}{\partial x} \right\rangle = \frac{\partial V}{\partial x}\bigg|_{x=\langle x \rangle}. \tag{3}$$

Clearly, equation (3) holds only when $\partial V/\partial x$ is a linear function of x, so that V must be a quadratic polynomial in x, possibly time-dependent:

$$V(x, t) = a(t) + b(t)x + c(t)x^2. \tag{4}$$

Obviously, three physically relevant cases are the free particle, the uniform field and the harmonic oscillator. Moreover, this result is readily extended to higher dimensions.

When the relation (4) is fulfilled, the Ehrenfest theorem allows a dramatic simplification of the quantum dynamical laws. As a matter of fact, we have passed from the Schrödinger equation (a partial differential equation in one spatial dimension plus the time), to a system of ordinary differential equations. We shall say that we have reduced the difficulty of our problem from dimension = 1 (Schrödinger), to dimension = zero (Ehrenfest).

In the next section, we shall derive from the Schrödinger equation a representation of QM which is based on a distribution function in the phase space (x, p). According to our definitions, such a representation has a dimension equal to two. These concepts will provide an interesting insight into the relationship between quantum, classical and statistical mechanics.

The Ehrenfest theorem is well known in the frame of QM. It might be interesting to show that the same result can be derived from the principles of classical statistical mechanics. The latter is based on the Liouville equation, which gives the evolution law for the probability distribution in the phase space $f(x, p, t)$

$$\frac{\partial f}{\partial t} + \frac{p}{m}\frac{\partial f}{\partial x} - \frac{\partial V}{\partial x}\frac{\partial f}{\partial p} = 0 \tag{5}$$

Let us multiply (5) first by x and integrate over x and p, then by p and integrate again. A little algebra shows that we obtain once again the system (2), where the mean value of a dynamical variable $A(x,p)$ is now defined by the following relation:

$$\langle A \rangle = \int A(x, p) f(x, p, t)\, \mathrm{d}x\, \mathrm{d}p \tag{6}$$

Once again, the system is not self-contained, except for the special family of quadratic potentials. In this latter case, the quantum mechanical and the classical descriptions for the mean values are strictly identical. If we introduce the force $E = -\partial V/\partial x$, the system reads, both in the classical and quantum cases, as:

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle x \rangle = \frac{\langle p \rangle}{m} \qquad \frac{\mathrm{d}}{\mathrm{d}t}\langle p \rangle = E(\langle x \rangle, t) \tag{7}$$

which is identical to the equations of motion in Newtonian dynamics.

The fact that the Ehrenfest relations can be derived both from QM and classical statistical mechanics should not, however, convey the idea that they express some 'profound' truth about physical phenomena. On the contrary, their physical meaning is, to a large extent, quite limited, and they can be at most considered as a formal property common to any statistical description of phenomena.

3. The Wigner picture of QM

As we have seen, the description of phenomena provided by classical statistical mechanics is based on the concept of phase space. The probability distribution $f(x, p, t)$ obeys the Liouville equation (5), and the average of any dynamical variable $A(x,p)$ is calculated as in equation (6). In QM, owing to the uncertainty principle, the very concept of phase space loses its meaning. In spite of this, in 1932 Wigner proposed a version of QM in which each quantum state is represented by a quasi-probability distribution in phase space that we shall hereafter designate as $W(x, p, t)$. If a 'good' definition of W is taken, the mean value of any dynamical variable can be calculated just as in the classical fashion, by making use of equation (6), where f is replaced by W. The 'good' definition for $W(x, p\, t)$ is the following:

$$W(x, p, t) = \frac{1}{2\pi\hbar}\int \exp\left(\frac{\mathrm{i}p\xi}{\hbar}\right)\Psi^*\left(x + \frac{\xi}{2}, t\right) \times \Psi\left(x - \frac{\xi}{2}, t\right) \mathrm{d}\xi \tag{8}$$

Moreover, $W(x, p, t)$ satisfies an equation that has some analogy with the Liouville equation (5):

$$\frac{\partial W}{\partial t} + \frac{p}{m}\frac{\partial W}{\partial x} = \frac{\mathrm{i}}{2\pi\hbar^2}\iint \mathrm{d}\lambda\, \mathrm{d}p' \exp \mathrm{i}\frac{\lambda}{\hbar}(p - p') \times \left\{ V\left(x - \frac{\lambda}{2}\right) - V\left(x + \frac{\lambda}{2}\right) \right\} W(x, p', t). \tag{9}$$

This is known as the Wigner or quantum Liouville equation, and it is directly derived from the Schrödinger equation through the definition (8).

Indeed, it would seem that the evolution equation (9) *plus* the two definitions (6) and (8) provide a frame in which QM can be treated on the same grounds as classical statistical mechanics. As a matter of fact, a few points need to be clarified.

(i) The Wigner functions, although it is always real, assumes negative values in nearly all cases, and consequently it cannot be interpreted as a classical joint probability distribution of x and p.

(ii) If, and only if, $\psi(x)$ is a Gaussian wavepacket, the Wigner function is non-negative for every x and p. Yet, even if a non-negative initial condition is chosen for equation (9), $W(x,p,t)$ will take unavoidably negative values after a certain time.

(iii) In order to represent a quantum pure state, the Wigner function must correspond to a wavefunction ψ through the formula (8). Nevertheless, in the treatment of statistical mixtures, we are often interested in functions $W(x,p)$ that do not derive from any ψ (it can be shown that a mixture is represented by the sum of Wigner functions of pure states). In this sense, one can state that the Wigner equation goes beyond the strict frame of the Schrödinger picture.

(iv) Since some terms of a dynamical variable $A(x,p)$ may not commute, it is necessary to establish a well determined, non-ambiguous correspondence rule between classical variables and quantum operators (Weyl's rule).

However, if one pays sufficient attention to the four points mentioned above, the Wigner function can be used operationally as a classical probability distribution to calculate mean values as in equation (6). In particular, by integrating $W(x,p)$ over p, we recover the usual quantum probability distribution for the position:

$$\int W(x,p)\,\mathrm{d}p = |\Psi(x)|^2$$

and, integrating over x, we find the probability distribution for the momentum

$$\int W(x,p)\,\mathrm{d}x = \frac{1}{\hbar}\left|\varphi\left(\frac{p}{\hbar}\right)\right|^2 \tag{10}$$

where φ is the Fourier transform of ψ.

The derivation of all the preceding formulae (8)–(10), as well as extended physical discussions, can be found in a review by Tatarskii (1983), and in the first chapters of Balescu (1975).

For the present purposes, we note that if the potential $V(x,t)$ is a quadratic polynomial in x as in (4), then the Wigner equation (9) becomes identical to the Liouville equation (5), as can be proved by directly substituting equation (4) in equation (9).

Therefore, for the class of potentials described by the relation (4), the evolution of any wavepacket can be treated in terms of classical statistical physics. On the other hand, all quantum features of the system are contained in its initial condition. In other words, for the particular case of a quadratic potential, we can construct a classical statistical ensemble that is equivalent to our quantum system. Notice however that the 'probability density' on such an ensemble can take negative values, and does not have therefore an immediate physical interpretation. The correspondence between a single quantum oscillator and an ensemble of classical oscillators was pointed out by Pippard (1983), by direct construction of the statistical ensemble: the Wigner approach is however more perspicuous insomuch as it immediately provides the probability density and its evolution equation.

The links between the different representations of QM that we mentioned in the previous section are now clear. On the one hand, when the potential is a quadratic polynomial, the Schrödinger equation can be reduced to a set of ODE (dimension = zero) for the centre of mass of the wavepacket, which thus follows Newtonian dynamics. On the other hand, it can be extended to a phase space formalism (dim = 2), which, under the same conditions, obeys the Liouville equation of classical statistical mechanics.

4. Particle in a uniform magnetic field

In this section we will generalize the preceeding results to the case of a charged particle moving in a uniform magnetic field. Although the proof is quite straightforward, this result is rarely mentioned in the literature on the Wigner equation (see, for example Canivell and Seglar (1978): they show that the non-negativity of a Wigner function is conserved if the Hamiltonian is quadratic (which includes the case of the uniform magnetic field): such a property will be a straightforward consequence of our results).

The magnetic field $\boldsymbol{B}$ is taken to be uniform and directed along the z axis: the motion is then bidimensional in the (x,y) plane. A suitable choice of the vector potential $\boldsymbol{A}$ is the following (Landau and Lifschitz 1969, section 111):

$$A_x = -By \qquad A_y = 0. \tag{11}$$

The canonical momentum $\boldsymbol{p}$ is defined, as usual, by

$$\boldsymbol{p} = m\boldsymbol{v} + e\boldsymbol{A} \tag{12}$$

(e and m are respectively the charge and the mass of the particle); the resulting Hamiltonian is

$$H = \frac{(\boldsymbol{p}-e\boldsymbol{A})^2}{2m} = \frac{p_x^2+p_y^2}{2m} + \tfrac{1}{2}m\omega^2y^2 + \omega y p_x \tag{13}$$

where $\omega = eB/m$ is the cyclotron frequency.

If our phase space is spanned by the four canonical variables x, p_x, y, p_y, the Liouville equation for $f(x,p_x,y,p_y)$ reads as:

$$\frac{\partial f}{\partial t} + \frac{p_x}{m}\frac{\partial f}{\partial x} + \frac{p_y}{m}\frac{\partial f}{\partial y} - m\omega^2 y\frac{\partial f}{\partial p_y} + \omega y\frac{\partial f}{\partial x} - \omega p_x\frac{\partial f}{\partial p_y} = 0. \tag{14}$$

From (13) the Schrödinger equation is clearly:

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\left(\frac{\partial^2\Psi}{\partial x^2} + \frac{\partial^2\Psi}{\partial y^2}\right) + \tfrac{1}{2}m\omega^2y^2\Psi - i\hbar\omega y\frac{\partial\Psi}{\partial x}. \tag{15}$$

In two dimensions, the Wigner function is defined as follows

$$W(x,y,p_x,p_y,t)=\frac{1}{(2\pi)^2}\iint\Psi\left(x-\frac{\xi}{2},y-\frac{\eta}{2},t\right)$$
$$\times\Psi^*\left(x+\frac{\xi}{2},y+\frac{\eta}{2},t\right)$$
$$\times\exp\mathrm{i}(\xi p_x+\eta p_y)\,\mathrm{d}\xi\mathrm{d}\eta. \qquad (16)$$

In equation (16) and in the following ones, we have taken, for simplicity, $\hbar=m=1$.

It is a matter of straightforward algebra, by means of the definition (16), to express each term of (14) through ψ and ψ^*. This results in:

$$\frac{\partial W}{\partial t}=\frac{1}{(2\pi)^2}\iint\left(\frac{\partial\Psi}{\partial t}\Psi^*+\Psi\frac{\partial\Psi^*}{\partial t}\right)G(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta$$

$$p_x\frac{\partial W}{\partial x}=\frac{1}{(2\pi)^2}\frac{\mathrm{i}}{2}\iint\left(\frac{\partial^2\Psi^*}{\partial x^2}\Psi-\Psi^*\frac{\partial^2\Psi}{\partial x^2}\right)G(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta$$

$$p_y\frac{\partial W}{\partial y}=\frac{1}{(2\pi)^2}\frac{\mathrm{i}}{2}\iint\left(\frac{\partial^2\Psi^*}{\partial y^2}\Psi-\Psi^*\frac{\partial^2\Psi}{\partial y^2}\right)G(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta \qquad (17)$$

$$\omega^2y\frac{\partial W}{\partial p_y}=\frac{1}{(2\pi)^2}\frac{\mathrm{i}\omega^2}{2}$$
$$\times\iint\left[\left(y+\frac{\eta}{2}\right)^2-\left(y-\frac{\eta}{2}\right)^2\right]\Psi\Psi^*G(\xi,\eta)\,\mathrm{d}\xi\mathrm{d}\eta$$

$$\omega\left(y\frac{\partial W}{\partial x}-p_x\frac{\partial W}{\partial p_y}\right)=\frac{\omega}{(2\pi)^2}$$
$$\times\iint\left[\left(y-\frac{\eta}{2}\right)\Psi^*\frac{\partial\Psi}{\partial x}+\left(y+\frac{\eta}{2}\right)\Psi\frac{\partial\Psi^*}{\partial x}\right]$$
$$\times G(\xi,\eta)\,\mathrm{d}\xi\mathrm{d}\eta$$

where

$$G(\xi,\eta)=\exp[\mathrm{i}(\xi p_x+\eta p_y)].$$

In relations (17) ψ is always understood to be evaluated at the point $(x-\xi/2,y-\eta/2)$, whereas ψ^* is evaluated at $(x+\xi/2,y+\eta/2)$. With the help of equation (15) and its complex conjugate it is easy to show that $W(x,p,t)$ obeys the Liouville equation (14). We have therefore proved that the motion of a charged particle in a uniform magnetic field can be expressed in terms of classical statistical mechanics, just as in the free particle, uniform force and harmonic oscillator cases.

In fact, *a posteriori*, this result is not too surprising, because the Hamiltonian (13) is still a quadratic form in x, y, p_x and p_y, although the proof is not trivial because of the mixed term yp_x.

5. Ehrenfest's theorem for a magnetic field

It might be interesting to show that the Ehrenfest's relations still hold for a charged particle moving in a magnetic field $\boldsymbol{B}(\boldsymbol{r})$, and that they reduce to a closed system of ODE (identical to the classical equations of motion) when the magnetic field is uniform (Zimmermann 1989). Again, these relations can be derived both from the Liouville and the Schrödinger equation.

Let us begin with the classical case. We write the Liouville equation in the more usual form, in which $f(\boldsymbol{r},\boldsymbol{v},t)$ is a function of the position $\boldsymbol{r}$ and the velocity $\boldsymbol{v}$, and the magnetic field is described by $\boldsymbol{B}$ rather than by the vector potential:

$$\frac{\partial f}{\partial t}+\boldsymbol{v}\cdot\frac{\partial f}{\partial\boldsymbol{r}}+\frac{e}{m}(\boldsymbol{v}\times\boldsymbol{B})\cdot\frac{\partial f}{\partial\boldsymbol{v}}=0. \qquad (18)$$

We multiply (18) first by $\boldsymbol{r}$ and integrate on $\boldsymbol{r}$ and $\boldsymbol{v}$, then by $\boldsymbol{v}$ and integrate again. After some integrations by parts, we obtain:

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\boldsymbol{r}\rangle=\langle\boldsymbol{v}\rangle\equiv\int\boldsymbol{v}f\,\mathrm{d}\boldsymbol{r}\mathrm{d}\boldsymbol{v}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\boldsymbol{v}\rangle=\frac{e}{m}\langle\boldsymbol{v}\times\boldsymbol{B}\rangle\equiv\frac{e}{m}\int\boldsymbol{v}\times\boldsymbol{B}f\mathrm{d}\boldsymbol{r}\mathrm{d}\boldsymbol{v}. \qquad (19)$$

The relations (19) are the analogue for a magnetic field of the Ehrenfest relations (2).

The second of equations (19) can be written as:

$$m\frac{\mathrm{d}\langle\boldsymbol{v}\rangle}{\mathrm{d}t}=\int\boldsymbol{J}\times\boldsymbol{B}\,\mathrm{d}\boldsymbol{r} \qquad (20)$$

where we have defined the current density:

$$\boldsymbol{J}(\boldsymbol{r},t)=e\int f(\boldsymbol{r},\boldsymbol{v},t)\boldsymbol{v}\,\mathrm{d}\boldsymbol{v}.$$

Equation (20) shows that the 'centre of mass' of the distribution (denoted by $\langle\boldsymbol{r}\rangle$ and $\langle\boldsymbol{v}\rangle$) moves under the action of a force which is the sum of all elementary Lorenz forces acting on each element of current $\boldsymbol{J}\,\mathrm{d}\boldsymbol{r}$.

When the magnetic field is uniform, it can be taken out of the integral in the second of equations (19), which become a closed system of ODE:

$$\frac{\mathrm{d}\langle\boldsymbol{r}\rangle}{\mathrm{d}t}=\langle\boldsymbol{v}\rangle \qquad \frac{\mathrm{d}\langle\boldsymbol{v}\rangle}{\mathrm{d}t}=\frac{e}{m}\langle\boldsymbol{v}\rangle\times\boldsymbol{B}. \qquad (21)$$

We now turn to the quantum mechanical case. The Heisenberg equations for the operators of position and velocity are given in Schiff (1985, section 24), for the case of a particle in a magnetic field $\boldsymbol{B}(\boldsymbol{r})$. They read

$$\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}=\boldsymbol{v} \qquad \frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t}=\frac{1}{2}\frac{e}{m}(\boldsymbol{v}\times\boldsymbol{B}-\boldsymbol{B}\times\boldsymbol{v}). \qquad (22)$$

Note that the two last terms are identical classically, but differ in QM, since $\boldsymbol{v}$ and $\boldsymbol{B}$ do not commute. In equation (22) $\boldsymbol{r}$, $\boldsymbol{B}$ and $\boldsymbol{A}$ are multiplication operators, while

$$\boldsymbol{v}=\frac{1}{m}(\boldsymbol{p}-e\boldsymbol{A})=\frac{1}{m}(-\mathrm{i}\hbar\nabla-e\boldsymbol{A}).$$

From equation (22) we can easily write the Ehrenfest relations for the mean values. Explicitly we have:

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle \boldsymbol{r}\rangle = \langle \boldsymbol{v}\rangle = \frac{1}{m}\int(-\mathrm{i}\hbar\Psi^*\nabla\Psi - e\boldsymbol{A}|\Psi|^2)\,\mathrm{d}\boldsymbol{r}$$

$$\begin{aligned} m\frac{\mathrm{d}}{\mathrm{d}t}\langle \boldsymbol{v}\rangle &= \frac{e}{2}\langle \boldsymbol{v}\times\boldsymbol{B} - \boldsymbol{B}\times\boldsymbol{v}\rangle \\ &= \frac{e}{2m}\int\Psi^*[(-\mathrm{i}\hbar\nabla - e\boldsymbol{A})\times\boldsymbol{B} \\ &\quad - \boldsymbol{B}\times(-\mathrm{i}\hbar\nabla - e\boldsymbol{A})]\Psi\,\mathrm{d}\boldsymbol{r} \\ &= -\frac{e^2}{m}\int|\Psi|^2\boldsymbol{A}\times\boldsymbol{B}\,\mathrm{d}\boldsymbol{r} \\ &\quad - \frac{\mathrm{i}\hbar e}{2m}\int\Psi^*[\nabla\times(\boldsymbol{B}\Psi) + \nabla\Psi\times\boldsymbol{B}]\,\mathrm{d}\boldsymbol{r}. \end{aligned} \tag{23}$$

We define, as usual, the current density (see Landau and Lifschitz 1969, Section 114, and also Feynman *et al* 1970, Section 21, for a deeper insight into the meaning of the current in the presence of a magnetic field):

$$\boldsymbol{J} = -\frac{\mathrm{i}\hbar e}{2m}(\Psi^*\nabla\Psi - \Psi\nabla\Psi^*) - \frac{e^2}{m}|\Psi|^2\boldsymbol{A}.$$

Using the previous definition, the second of equations (23) becomes:

$$m\frac{\mathrm{d}\langle \boldsymbol{v}\rangle}{\mathrm{d}t} = \int\boldsymbol{J}\times\boldsymbol{B}\,\mathrm{d}\boldsymbol{r} \tag{24}$$

which is formally identical to the classical formula (20).

Once again, when $\boldsymbol{B}$ is uniform, the equations (23) for the mean values become a closed system of ODE, identical to equation (21).

In summary, in this section we have extended the Ehrenfest relations to the case of a magnetic field. The results are in accordance with what had been found in the previous paragraphs. Moreover, they can be easily generalized to the most general case of a time-dependent electromagnetic field.

4. A numerical simulation

In order to get a visual picture of the motion of a charged particle in a uniform magnetic field, we have solved numerically the bidimensional Schrödinger equation (15). The initial condition is a localized Gaussian wavepacket:

$$\Psi_0(x,y) = \left(\frac{1}{2\pi\sigma_x\sigma_y}\right)^{1/2} \times \exp-\left(\frac{(x-x_0)^2}{4\sigma_x^2} + \frac{(y-y_0)^2}{4\sigma_y^2}\right). \tag{25}$$

It should be noted that (25) represents a wavepacket for which the mean value of the *canonical* momentum $\boldsymbol{p}$ is zero. Yet the mean values of the x and y components of the velocity are (see equation (12)):

$$\langle v_x\rangle = \frac{1}{m}(\langle p_x\rangle - e\langle A_x\rangle) = \frac{eB}{m}y_0 = \omega y_0$$

$$\langle v_y\rangle = \frac{1}{m}(\langle p_y\rangle - e\langle A_y\rangle) = 0. \tag{26}$$

where we have made use of equations (11) and (25).

A classical particle with initial position (x_0, y_0) and initial velocity given by (26), immersed in a uniform magnetic field parallel to the z axis, will rotate with constant angular velocity ω around the point $(x_0, 0)$, describing a circle of radius $R = |y_0|$.

According to the previous theorem, the 'centre of mass' of the quantum wavepacket must describe the same circle as the classical particle. The wavefunction will be deformed during the evolution, but it must come back exactly to its initial condition after one period $T = 2\pi/\omega$.

In the numerical simulation we have chosen $x_0 = 0$, so that the particle turns around the origin. The other parameters are:

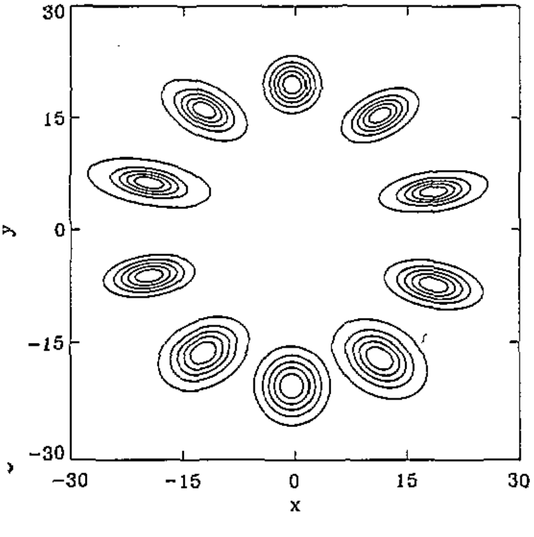
$$\hbar = m = 1 \qquad y_0 = 20$$

$$\omega = 0.1\pi \qquad (T = 20)$$

$$\sigma_x = 2 \qquad \sigma_y = 2.$$

Figure 1 gives the time evolution of the density $|\Psi(x,y)|^2$ at time intervals $\Delta t = 2$. Each picture is formed by five isodensity contour levels. Indeed, we recover the classical motion, and the packet comes back to the initial condition after one period.

Figure 1. Evolution with time of a wavepacket in a uniform magnetic field. Each picture represents the spatial density, at time intervals of $\Delta t = 2$.



6. Conclusion

Thousands of papers have been written on the ultimate structure of QM, its relation to classical physics, a possible statistical interpretation, the meaning of phase space, etc. The reader can legitimately ask the reasons for this new one. Let us attempt its justification.

Most of the existing literature deals with stationary states, and focuses on the eigenvalues and eigenfunctions—usually to interpret spectroscopic data. It is indeed in this domain that the standard theory of QM has proved to work extremely well. A much more limited number of papers focuses on time-dependent situations, and among them very few mention the Wigner function and Wigner equation. Due to the fact that it introduces negative probabilities (and also, perhaps, because of a too strict application of the Copenhagen school ideas), many authors are reluctant to consider it other than a curious mathematical object, possessing very little physical meaning, if any.

In our opinion, two reasons render the Wigner representation particularly interesting.

First, when dealing with pure states, it is strictly equivalent to the Schrödinger representation, provided the initial condition can be derived from a wavefunction through the formula (8). For a wider class of initial conditions, the Wigner equation provides the evolution of a statistical mixture. In fact, the possibility to treat both pure and mixed states via the same evolution equation seems to us an appealing property of the Wigner representation.

Secondly, the mathematical properties of the Wigner function make it an ideal tool to investigate the classical limit of QM. Instead of worrying about the negative probabilities it introduces, an interesting challenge would be to interpret their meaning by a suitable generalization of the traditional probability theory. Our present goals were however more modest and, as we said before, more technical, concentrating on the links between the different representations. They are resumed on the scheme. We distinguish three types of representations, and require each of them to be self-contained.

At the lowest (simplest) level of dimension zero, we have an ODE describing either the deterministic motion of a classical particle or the motion of the centre of mass of the wavepacket for the quadratic Hamiltonians (free particle, uniform force, harmonic oscillator and uniform magnetic field).

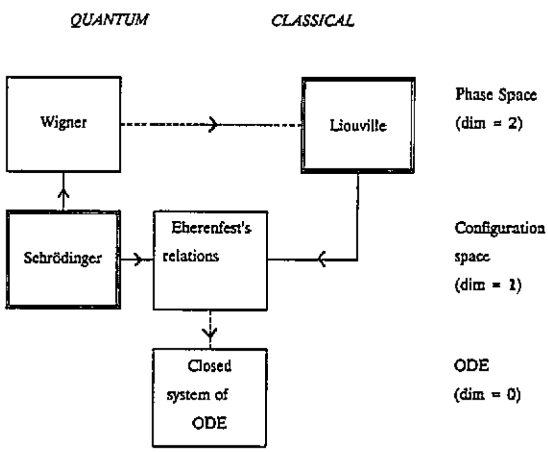
At the next level (dimension one) we get the Schrödinger equation, fundamental in QM, which resumes, by the aid of just one coordinate, our information on both position and momentum. It has no classical counterpart.

The Ehrenfest relations are a step toward the lower level, but in the general case they are void of meaning, since to apply them one needs the knowledge of the wavefunction. On the other hand, these formal relations are also obtained in the classical case, when a statistical approach has to be used either to describe many particles or our ignorance about the initial state of one particle.

At the highest level (phase space, dimension two), we find the Liouville equation for classical systems and the Wigner equation for quantum-mechanical ones (both pure states and mixtures). When the Hamiltonian is a quadratic form, these two equations coincide, and all the difference between classical and quantum physics is contained in the initial condition (arbitrary in the former case, but not in the latter).

Consequently, the class of quadratic Hamiltonians displays two interesting properties. First, it allows us to construct a self-contained ODE for the centre of the wavepacket, which behaves as a classical particle. On the other hand, the details of the wavefunction can be obtained by considering a classical phase space problem, in which the probability density, although involving negative values, is invariant along the classical trajectories.

(In the scheme, the double boxes stand for the most fundamental equations. A full line means implication in general, whereas a broken line means that the implication is valid only for the special class of quadratic Hamiltonians).



Acknowledgments

The authors thank the Associazione Sviluppo Scientifico e Tecnologico del Piemonte, Turin, Italy, for financial support.

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