Rescaling methods and plasma expansions into vacuum

G. Manfredi, S. Mola, and M. R. Feix
PMMS/CNRS, 3A, Avenue de la Recherche Scientifique, 45071 Orléans Cedex 02, France

(Received 22 April 1992; accepted 25 September 1992)

The problem of a two-component, collisionless plasma expansion into vacuum is investigated from the viewpoint of the Vlasov–Poisson model. The set of equations is treated both analytically (through the rescaling transformations) and numerically, using a one-dimensional Eulerian code. In planar geometry, the rescaling allows to conjecture the existence of a self-similar expansion over long times. Numerical results subsequently confirm the conjecture and show that the plasma becomes neutral over a smaller and smaller scale. A few thermodynamical properties are studied: the temperature is shown to decrease as t^{-2}; the polytropic relation $(d/dt)(pn^{-\gamma})=0$ (with $\gamma=3$) is verified asymptotically via a semianalytical argument. Finally, the same problem is studied in a spherical one-dimensional geometry. The time-asymptotic solution is again self-similar. Numerical simulations show that a non-neutral, multiple-layer structure appears, which is proved to be stable over long times.

I. INTRODUCTION

The expansion of a plasma into vacuum plays an important role in many areas of plasma physics, and it has received a great deal of attention since the early days of this discipline. The main phenomenon associated with plasma expansion is the acceleration of positive ions to supersonic velocities, a process that has been observed in laboratories since the 1930s by investigators working with vacuum arc experiments.[1,2] In 1961, Plyutto[3] first recognized that the process of ion acceleration was a direct consequence of the plasma expansion into vacuum. According to Plyutto, the lighter and more mobile electrons tend to run ahead the bulk of the plasma, creating a self-consistent electric field that accelerates the ions to high velocities. The explanation given by Plyutto is important, because it interprets the process of ion acceleration in terms of exclusively electrostatic phenomena. Since then, plasma expansions have been investigated theoretically and numerically using purely electrostatic models (Poisson's law) associated to a set of hydrodynamic or kinetic equations.

Plasma expansion has been proposed as an important phenomenon taking place in a great deal of areas, ranging from astrophysics to nuclear fusion. In 1969, plasma expansion was suggested to be related to the problem of the interaction of terrestrial plasma with rapidly moving objects such as satellites.[4] In the works of Singh and Schunk,[5] an explanation of polar wind phenomena was given in terms of plasma expansion.

In fusion technology, plasma expansion occurs in inertial confinement experiments. A laser beam heats and ionizes a solid pellet, generating a high-density plasma that rapidly expands outward.[6,7]

During the last few years, a great deal of laboratory experiments have been devoted to the investigation of plasma expansion into a vacuum.[8,9] Usually, in such experiments, the plasma expands along the magnetic field lines of an external uniform magnetic field directed along the z axis. The magnetic field prevents particles from expanding along the x and y directions; the experimental setup can be arranged in order to realize a realistic one-dimensional expansion.

More recent experiments deal with particular non-Maxwellian velocity distribution for the electrons (Hairapetian and Stenzel[10]).

The great majority of theoretical and numerical works on plasma expansion has been done in the framework of the hydrodynamic model.[11–13] An excellent review on this subject has been published in 1987 by Sack and Schamel,[14] and, since, few new results have been achieved. In the following, we shall briefly summarize the state of the art in the theoretical and numerical investigation of plasma expansion. We shall postulate a one-dimensional, planar geometry, with no magnetic field (either external or self-consistent). The plasma consists of single charged ions of mass m_i and charge $+e$ and electrons of mass m_e and charge $-e$. Two assumptions on the dynamics of ions and electrons are made: (a) The ion temperature is negligible compared to the electron temperature ($T_i/T_e\to 0$). Consequently, the ion pressure can be neglected. (b) The electron mass is negligible compared to the ion mass ($m_e/m_i\to 0$). Consequently, the electrons can be considered in thermal equilibrium.

The set of fluid equations reads as

$$\frac{\partial n_i}{\partial t}+\frac{\partial}{\partial x}(n_i v_i)=0, \tag{1a}$$

$$\frac{\partial v_i}{\partial t}+v_i\frac{\partial v_i}{\partial x}=-\frac{e}{m_i}\frac{\partial \phi}{\partial x}, \tag{1b}$$

$$n_e=n_0\exp(e\phi/kT_e). \tag{1c}$$

The potential ϕ is given by the Poisson equation:

$$\frac{\partial^2\phi}{\partial x^2}=\frac{e}{\epsilon_0}(n_e-n_i). \tag{2}$$

 0899-8221/93/020388-14$06.00

Here, n_i and n_e are the densities of the two species, v_i is the ion velocity, k is the Boltzmann constant, and ϵ_0 is the dielectric constant in vacuum.

Equation (1c) postulates a constant temperature for the electron population, which is a reasonable assumption when the plasma is continuously generated at the source and its thermal conductivity is high.

In more general circumstances this may not be true, and some authors assume a polytropic equation of state,

$$p_e = c n_e^{\gamma}, \tag{3}$$

where p_e is the electron pressure, c is a constant, and γ is the polytropic exponent ($\gamma=1$ in the isothermal case). When (3) holds, the electron density dependence on the potential is given by the following equation:[14]

$$n_e = n_0 \left(1 + \frac{\gamma-1}{\gamma} \cdot \frac{e\phi}{kT_e}\right)^{1/(\gamma-1)}; \quad \gamma \neq 1. \tag{1c'}$$

Equations (1) and (2) form a closed set of evolution equations that can be solved with suitable initial and boundary conditions. Unfortunately, analytic solutions are not available for the most general case. Yet, if one makes the additional assumption of charge neutrality ($n_e = n_i = n$), the Poisson equation (2) is no more useful and the system (1) and (2) reduces to the following:

$$\begin{aligned} &\frac{\partial n}{\partial t} + \frac{\partial}{\partial x}(nv) = 0, \\ &\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial x} = -\gamma n^{\gamma-2} \frac{\partial n}{\partial x}, \end{aligned} \tag{4}$$

where we have used normalized units.

The system (4) possesses a self-similar solution, in which all the quantities are functions of the self-similar variable:

$$\tau = x/t. \tag{5}$$

The set of self-similar solutions reads, for $\gamma \neq 1$ (Sack and Schamel[15]) as

$$\begin{aligned} n(\tau) &= \left(1 - \frac{\gamma-1}{\sqrt{\gamma}(\gamma+1)}(\tau + \sqrt{\gamma})\right)^{2/(\gamma-1)}, \\ v(\tau) &= \frac{2}{\gamma+1}(\tau + \sqrt{\gamma}), \end{aligned} \tag{6}$$

and for $\gamma = 1$ (Gurevich *et al.*[11] and Allen *et al.*[12]):

$$\begin{aligned} n(\tau) &= \exp - (\tau + 1), \\ v(\tau) &= \tau + 1. \end{aligned} \tag{7}$$

The range of τ is restricted by the following inequality:

$$-\sqrt{\gamma} < \tau < 2\sqrt{\gamma}/(\gamma - 1).$$

For $\tau < -\sqrt{\gamma}$, n is constant and equal to unity, while v is zero. For $\tau > 2\sqrt{\gamma}/(\gamma-1)$ both n and v are zero.

The solutions (6) and (7) represent a rarefaction wave that propagates at sound speed from the interface plasma–vacuum toward the bulk of the unperturbed plasma.

When we abandon the assumption of quasineutrality and take into account charge separation effects, the Poisson equation can no more be neglected and we have to solve the entire system (1) and (2). In this case, self-similar solutions are not possible and one has to resort to numerical calculations.

Extensive numerical results are reported in Sack and Schamel,[14,15] both for the charge separation case [system (1) and (2)] and for the quasineutrality case [system (4)]. Each case was studied for different values of the polytropic exponent γ. Their simulations show that the ion density soon develops a spikelike structure at its front. Such structure quickly grows up, getting sharper and sharper, and eventually the numerical solution collapses. The authors exclude that such a collapse is due to numerical instabilities.

Since the breaking down of the solution is observed for every value of $\gamma (1 \leq \gamma \leq 2)$, both in the quasineutrality and charge separation cases, its occurrence should not be due to either the assumption of isothermality or charge neutrality, but rather, in our opinion, to the more profound question of the validity of the hydrodynamic approximation.

From the previous results, it is quite clear that the hydrodynamic model is not adequate to describe the plasma expansion into vacuum. In fact, strictly speaking, a hydrodynamic treatment of a collisionless plasma is, in principle, impossible. This is due to the fact that the relation $pn^{-\gamma} = \text{const}$ does not derive from the Vlasov equation, but is imposed *ad hoc* in order to close the set of equations. For a general phase space distribution $f(x,v,t)$, such a relation is by no means satisfied, except in a few very special cases. Our Vlasov treatment will tell us when and how the hydrodynamic approximation is a correct one. We shall see that the polytropic relation must be written in the more general form:

$$\frac{d}{dt}(pn^{-\gamma}) = \left(\frac{\partial}{\partial t} + v \frac{\partial}{\partial x}\right)(pn^{-\gamma}) = 0,$$

with, imperatively, $\gamma = 3$ (see Sec. V).

The point of view adopted in this paper is to keep a collisionless regime, using a kinetic model to describe the dynamics of the plasma. In particular, we shall treat both the ions and the electrons dynamics through two Vlasov equations, coupled by the Poisson equation. Works concerned with the numerical solution of Vlasov models of the plasma–vacuum system are still very limited in number (see Denavit[16] and Galvez and Borowski[17]), and not completely satisfactory. In particular, to our knowledge, no solutions over long times have ever been produced. This is due principally to two intrinsic difficulties of the numerical treatment, namely the following.

(1) Previous simulations have been performed using particle codes, which, as it is well known, exhibit considerable numerical noise at low densities. Since the interesting phenomena (such as the steepening of the ion density) occur precisely in a region of low density, such codes are not quite suited for this kind of problem.

(2) A true expansion-into-vacuum problem requires free boundary conditions. However, this is not trivial, since the region occupied by the plasma becomes larger and larger with time. In previous works, the authors have used either periodic or absorbing boundaries, in which a particle reaching the boundaries is simply removed from the system. Obviously, such a treatment alters the particle distribution at the plasma front and, eventually, corrupts the entire solution over long times.

The aim of this work is to remove both the abovementioned restrictions.

As to point (1), we resort to codes solving the Vlasov equation by direct discretization of the phase space (Eulerian codes). Such codes are certainly more time and memory consuming than the ordinary particle codes, but exhibit much less numerical noise, and allow a fine resolution of phase space structures, even in regions of low density. In particular, for one-dimensional problems, the numerical effort is not prohibitive (most of the calculations have been performed on Sun workstations), and the use of Eulerian codes is highly recommended.

The second point is treated via the so-called rescaling transformations,[18–22] which will be analyzed in detail in the next sections. The philosophy of the rescaling technique consists of introducing new space and time variables, so that the expansion term in the solution is automatically taken into account by the transformation. As a result, in the new variables, the plasma experiences no more expansion, and the free boundary conditions can be easily imposed.

It is important to point out that the rescaling is not simply a numerical technique. In fact, it very often allows to conjecture the structure of the asymptotic solution. The numerical work subsequently checks whether the conjecture is right, and provides the details of the conjectured structure. This double aspect, analytical and numerical, seems to us important in developing new computational tools. To our knowledge, this paper is the first to introduce the numerical aspect of rescaling in plasma physics.

These methods allow us to obtain the solution over long times. In the one-dimensional planar case, we show that the neutral self-similar solution is approached asymptotically. The plasma expands then freely as a neutral gas (ballistic motion). The phase portrait shows that the space and velocity variables are highly correlated by the relation $v=x/t$. We also show that the temperature of the plasma decreases, in the asymptotic regime, as t^{-2}. Such a result invalidates *a priori* the hypothesis of isothermality, which is often assumed in the hydrodynamic models. Via a simple calculation in the rescaled space, it will be shown that the latter are of some validity under very special conditions, including the existence of self-similar solutions.

Subsequently, we shall study the plasma expansion in the case of a one-dimensional spherical geometry, which, to our knowledge, no author has so far investigated. It will turn out that, although the time-asymptotic solution is again self-similar, the plasma does not approach neutrality. Contrarily, a stable double-layers structure appears, indicating that the ions and the electrons are totally decoupled. This result contributes to confirm that double layers can be generated and persist, even in collisionless systems.

II. MATHEMATICAL MODEL

We consider a one-dimensional, collisionless, two-species plasma in planar geometry, which is freely expanding into vacuum, experiencing no external field. The two species have identical electric charge ($+q$ and $-q$) but different masses. The evolution of the plasma can be described by the Vlasov–Poisson system:

$$\begin{aligned}&\frac{\partial f_e}{\partial t}+v\frac{\partial f_e}{\partial x}-E\frac{\partial f_e}{\partial v}=0,\\&\frac{\partial f_i}{\partial t}+v\frac{\partial f_i}{\partial x}+\frac{E}{M}\frac{\partial f_i}{\partial v}=0,\\&\frac{\partial E}{\partial x}=\int(f_i-f_e)dv,\end{aligned}\tag{8}$$

where $f_e(x,v,t)$, $f_i(x,v,t)$ are the phase space distribution functions for electrons and ions, respectively. To simplify the notations we have taken,

$$q_i=-q_e=m_e=\epsilon_0=1;\quad M=m_i/m_e.$$

Initially the plasma is described by $f_{i,e}(x,v,t=0)$, having finite support in the phase space. Our purpose is to determine the time-asymptotic solution of the system (8). In particular, we want to answer the following questions: (a) Does the plasma become progressively neutral over smaller and smaller distances? (b) Or contrarily, does it form compact sheets of alternatively negative and positive net charge? (c) Is the expansion law of the "Hubble form": $v\sim x/t$?

The system (8) is usually solved numerically, taking periodic or absorbing boundary conditions in the x coordinate. On the contrary, as it was pointed out before, one should use free boundary conditions, and then integrate (8) on an interval $0\leqslant x\leqslant L$ large enough to contain the plasma until it reaches the asymptotic solution. If we want to follow the evolution for relatively long times, we have to discretize large intervals with a great number of mesh points, and the numerical effort soon becomes prohibitive. In order to avoid this difficulty—and also to obtain some information on the structure of the asymptotic solution—we do some preliminary analytical work on the system (8).

III. RESCALING METHODS

In previous works, the rescaling methods have been mostly used as an analytical tool in the study of nonlinear ordinary and partial differential equations. Although the method is somewhat of an extension of the self-similarity analysis,[23] it does not introduce any limitation in the choice of the initial conditions, the transformed equations being strictly equivalent to the original ones. Some analytical applications of rescaling can be found in the study of nonlinear diffusion,[18] the Vlasov–Poisson system,[19] the Schrödinger equation,[20] and other evolution problems.[21,22]

In the following paragraphs, we shall illustrate the advantages of the rescaling technique as an analyticocomputational tool.

Let us introduce a "new rescaled space time" (ξ,θ), defined by

$$x=C(t)\xi, \qquad dt=A^2(t)d\theta. \tag{9}$$

We want to introduce a "new rescaled phase space" (ξ,η). In order to do so, the "new velocity" η must be defined, as usual, as the derivative of ξ with respect to θ, with the following relation between the old and new velocity:

$$v=\frac{dx}{dt}=\eta\,\frac{C}{A^2}+\xi\dot{C} \tag{10}$$

(the overdot indicates derivation with respect to t).

Finally, we rescale also the dependent variables f_i, f_e, and E:

$$f_{e,i}(x,v,t)=G(t)F_{e,i}(\xi,\eta,\theta), \qquad E(x,t)=H(t)\epsilon(\xi,\theta). \tag{11}$$

Taking into account (8)–(11) we obtain a "new Vlasov–Poisson system" for the rescaled quantities,

$$\frac{\partial F_i}{\partial\theta}+\eta\,\frac{\partial F_i}{\partial\xi}+2A^2\left(\frac{\dot{A}}{A}-\frac{\dot{C}}{C}\right)\eta\,\frac{\partial F_i}{\partial\eta}+\frac{\epsilon}{M}\,\frac{HA^4}{C}\,\frac{\partial F_i}{\partial\eta}-\frac{A^4\ddot{C}}{C}\,\xi\,\frac{\partial F_i}{\partial\eta}+\frac{\dot{G}A^2}{G}\,F_i=0,$$

$$\frac{\partial F_e}{\partial\theta}+\eta\,\frac{\partial F_e}{\partial\xi}+2A^2\left(\frac{\dot{A}}{A}-\frac{\dot{C}}{C}\right)\eta\,\frac{\partial F_e}{\partial\eta}-\epsilon\,\frac{HA^4}{C}\,\frac{\partial F_e}{\partial\eta}-\frac{A^4\ddot{C}}{C}\,\xi\,\frac{\partial F_e}{\partial\eta}+\frac{\dot{G}A^2}{G}\,F_e=0, \tag{12}$$

$$\frac{\partial\epsilon}{\partial\xi}=\frac{GC^2}{HA^2}\int(F_i-F_e)d\eta.$$

Though the system (12) is rather complicated, we still dispose of four arbitrary functions $A(t)$, $C(t)$, $G(t)$, and $H(t)$, subjected to the sole constraint to be regular and nonzero over $[0,\infty[$.

A first requirement is that the first and second equations of (12) have the form of the Liouville equation for a system with friction, which reads as

$$\frac{\partial F}{\partial\theta}+\eta\,\frac{\partial F}{\partial\xi}+\frac{\partial}{\partial\eta}\,[\Gamma(\eta)F]=0, \tag{13}$$

where Γ is a force depending on the velocity η. Equation (13) imposes that

$$\frac{\dot{G}}{G}=2\left(\frac{\dot{A}}{A}-\frac{\dot{C}}{C}\right),$$

which is immediately integrated to give

$$GC^2/A^2=\text{const}=1. \tag{14}$$

The relation (14) can be written, taking into account (9)–(11),

$$f_{i,e}\,dx\,dv=F_{i,e}\,d\xi\,d\eta$$

indicating the conservation of the number of electrons and ions in the two spaces. Moreover, if we choose $H=C/A^4$, the electric field is left invariant:

$$\epsilon(\xi,\theta)=E(x,t), \tag{15}$$

while the Poisson equation becomes

$$\frac{\partial\epsilon}{\partial\xi}=\frac{1}{H(t)}\int(F_i-F_e)d\eta.$$

We note that, in the rescaled space, the factor $H(t)$ plays the role of a "time-dependent dielectric constant."

When (14) and (15) are satisfied, the force term Γ in the rescaled Vlasov equation (13) has the form

$$\begin{aligned}\Gamma=&\ \epsilon && \text{rescaled field}\\ &-\frac{A^4\ddot{C}}{C}\,\xi && \text{"transformation field"}\\ &+2A^2\left(\frac{\dot{A}}{A}-\frac{\dot{C}}{C}\right)\eta, && \text{friction.}\end{aligned} \tag{16}$$

We note that some "unusual" terms appear in the rescaled force; namely a linear "transformation field" (confining for both species if $\ddot{C}>0$, i.e., if $C(t)$ goes to infinity faster than t) and a friction term. Let us underline that for $A(t)=C(t)$ the friction disappears and we have

$$dx\,dv=d\xi\,d\eta, \tag{17}$$

i.e., the phase space volume element is conserved.

The philosophy of the rescaling methods consists in interpreting the transformed Vlasov equations (13) and (16) as describing a "new" physical system, in which the particles experience an electrostatic interaction via the Poisson law, an external force (the transformation field), and a dissipative term (friction).

It should be stressed that the law of asymptotic evolution is entirely described by the factor $C(t)$. If we could guess the "right" law of expansion, then, owing to (9), we would have $\xi=$const: in other words, in the space (ξ,η) the plasma would experience no expansion. From a numerical point of view, this is an important result, since, with a suitable adjustment of the transformation parameters, we can "freeze" the support of $F(\xi,\eta,\theta)$ in ξ on a finite interval $[0,L]$. It is easy to understand that a regular mesh $\Delta\xi$ discretizing the fixed interval $[0,L]$ is equivalent to a "moving mesh" $\Delta x=C(t)\Delta\xi$ on the interval with moving boundary $[0,C(t)L]$.

Here, we formulate the conjecture that the asymptotic law of expansion is of the form $x\sim t$; in other words, we suppose that the plasma behaves asymptotically as a neutral gas. Numerical results will confirm our conjecture and precisely how the plasma approaches neutrality.

Our conjecture imposes the following choice:

$$C(t)=1+\Omega t, \tag{18}$$

where Ω is an arbitrary frequency, characterizing the transformation.

The friction terms are numerically difficult to deal with, since they usually lead to Dirac's delta functions in the solution. (This is because friction eventually brings all velocities to zero. Note that a friction term can sometimes be useful to infer intuitively an asymptotic solution,[19] just as one can easily predict the final state of a damped pendulum.) In order to avoid friction, we take

$$A(t)=C(t)=1+\Omega t. \tag{19}$$

Then the condition (14) requires $G(t)=1$.

The relation between the old and the new velocity becomes, from Eq. (10),

$$v=\eta/(1+\Omega t)+\Omega\xi. \tag{20}$$

Finally, the rescaled Vlasov–Poisson system reads as

$$\begin{aligned}&\frac{\partial F_i}{\partial\theta}+\eta\,\frac{\partial F_i}{\partial\xi}+\frac{\epsilon}{M}\,\frac{\partial F_i}{\partial\eta}=0,\\ &\frac{\partial F_e}{\partial\theta}+\eta\,\frac{\partial F_e}{\partial\xi}-\epsilon\,\frac{\partial F_e}{\partial\eta}=0,\\ &\frac{\partial\epsilon}{\partial\xi}=\frac{1}{(1-\Omega\theta)^3}\int(F_i-F_e)d\eta.\end{aligned} \tag{21}$$

Integrating the second equation of (9), we obtain

$$1-\Omega\theta=(1+\Omega t)^{-1}, \tag{22}$$

which indicates that the time θ is "renormalized" on a finite interval. In fact,

$$0\leqslant\Omega\theta\leqslant 1.$$

The time $\theta=1/\Omega$ is then a singular point on the right-hand side of the Poisson equation. This may cause numerical difficulties, requiring taking smaller and smaller time steps $\Delta\theta$ as θ approaches $1/\Omega$. Anyway, the use of a slightly different rescaling scheme, which will be shown in the next paragraph, allows us to overcome this problem.

First, let us give a physical interpretation of (21). It represents a system in which the "dielectric constant" tends to zero as $\Omega\theta\to 1$. Consequently, also, the Debye length tends to zero in the rescaled space, indicating that non-neutral regions can exist asymptotically on a smaller and smaller scale. Moreover, the time θ being limited to the value $1/\Omega$, the system cannot experience an infinite displacement, and is then "frozen" on a finite interval.

Numerical results (Sec. V) will subsequently confirm these conjectures.

IV. RESCALING WITH "VARIABLE MASS"

Let us introduce the new independent variable,

$$\pi=m_{i,e}\mu(t)\eta=m_{i,e}(C^2/A^2)\eta. \tag{23}$$

We shall call $m_{i,e}\mu(t)$ the (time-dependent) "mass" and π the "momentum" for obvious reasons of analogy with the relation $p=mv$.

From Eq. (10), one easily finds the relation between the old and the new momentum:

$$p=m_{i,e}v=\pi/C+\xi\dot{C}m_{i,e}. \tag{24}$$

Note that (24) does not depend anymore on A. The relation (24) between p and π is the same (apart from the constant factor $m_{i,e}$) as the relation (10) between v and η, when $A=C$.

In fact, the phase space element is now conserved for every choice of A and C:

$$d\pi\, d\xi=dp\, dx.$$

This is due to the fact that Eqs. (9)–(24) constitute a canonical transformation, which conserves the Hamiltonian formalism. The old and the new Hamiltonians are given by (for electrons with $m_e=1$):

$$H_{\text{old}}(x,p,t)=\frac{p^2}{2}+V(x,t);\quad E=-\frac{\partial V}{\partial x},$$

$$H_{\text{new}}(\xi,\pi,\theta)=\frac{\pi^2}{2\mu(\theta)}+A^2C\left(\frac{1}{2}\,\ddot{C}\xi^2+\Phi(\xi,\theta)\right);$$

$$\epsilon=-\frac{\partial\Phi}{\partial\xi}.$$

The conservation of the number of particles now imposes

$$F(\xi,\pi,\theta)=f(x,p,t).$$

Moreover, we now want to keep the form of the Poisson equation. In order to do so, we have to choose

$$\epsilon(\xi,\theta)=E(x,t).$$

Finally, the Vlasov equation in the space (ξ,π) reads as

$$\frac{\partial F}{\partial\theta}+\frac{A^2}{C^2}\,\pi\,\frac{\partial F}{\partial\xi}+A^2C(\epsilon-\ddot{C}\xi)\,\frac{\partial F}{\partial\pi}=0, \tag{25}$$

while the Poisson equation is left invariant. The system being Hamiltonian, the friction term has disappeared.

Dividing Eq. (25) by A^2, and taking into account the second of (9), we may come back to the "old" time variable t. The factor $A(t)$ then disappears from Eq. (25). This operation shows that the roles of $A(t)$ and $C(t)$ are totally decoupled, and therefore they can be chosen independently. From a numerical point of view, we have gained one more degree of freedom in the choice of the functions characterizing the transformation. In particular, the factor $C(t)$ determines the structure of the phase space (ξ,π), while $A(t)$ determines the time scale. We shall therefore take $C(t)$ so as to absorb the asymptotic term of the expansion. As it was done before, we choose for the case of a planar expansion: $C(t)=1+\Omega t$.

With this choice, Eq. (24) takes the following form:

$$p=\pi/(1+\Omega t)+m_{i,e}\Omega\xi. \tag{26}$$

On the other hand, $A(t)$ must be selected according to the characteristic time scale of the problem, which is given by the plasma frequency ω_p.

If we take $A^2(t)=\omega_p^{-1}(t)$, we have

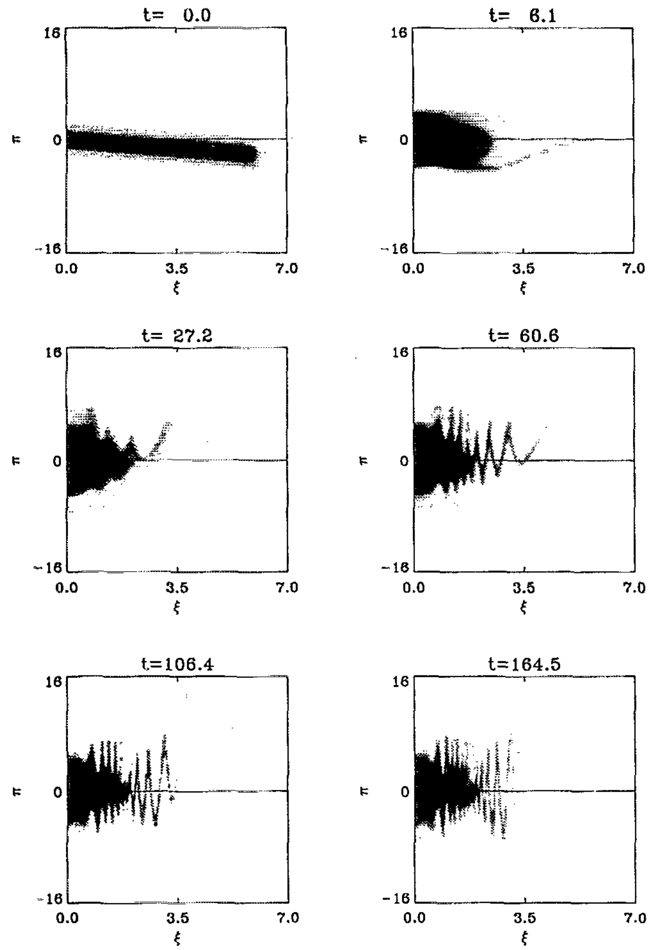

FIG. 1. Electron distribution function in the rescaled phase space (planar geometry). Dark regions represent regions of high density.

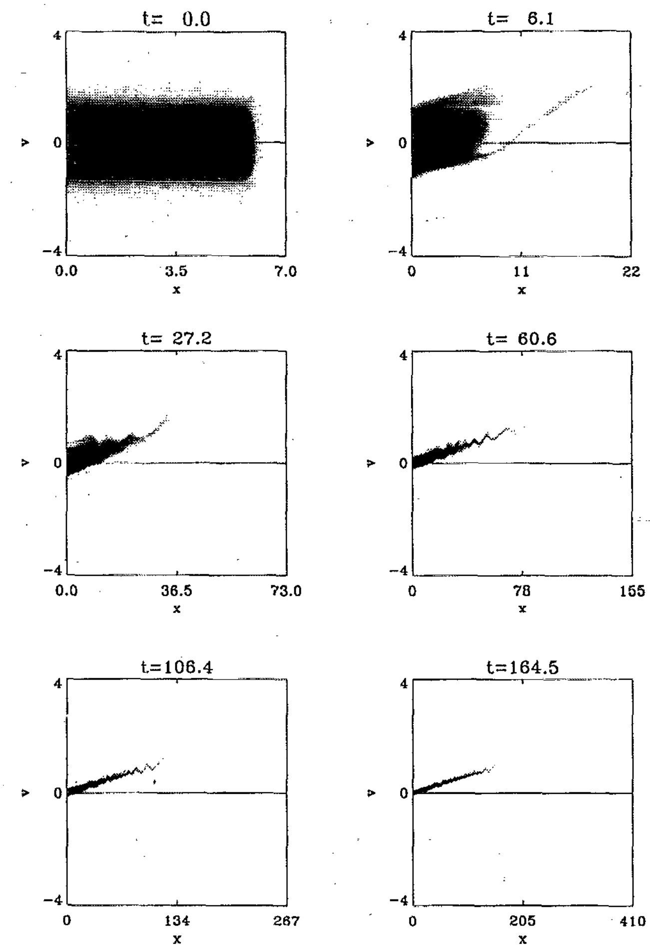

FIG. 2. Electron distribution function in the real phase space (planar geometry).

$$d\theta=\frac{dt}{A^2}=\frac{dt}{\omega_p^{-1}}.$$

With this choice, and taking constant time steps $\Delta\theta$ in the numerical integration, the real time t is automatically sampled in units of ω_p^{-1}. Since $\omega_p \propto \sqrt{n} \propto C^{-1/2}$, $n(x)$ being the spatial density, we have

$$A^4=C=1+\Omega t.$$

Integration of the second of (9) gives, for this choice of $A(t)$,

$$1+\Omega t=(1+\Omega\theta/2)^2,$$

and for $t\to\infty$ also $\theta\to\infty$.

Finally, the Vlasov–Poisson system for a two-component plasma becomes

$$\frac{\partial F_i}{\partial\theta}+\frac{\pi}{M\mu(\theta)}\frac{\partial F_i}{\partial\xi}+\mu(\theta)\epsilon\frac{\partial F_i}{\partial\pi}=0,$$

$$\frac{\partial F_e}{\partial\theta}+\frac{\pi}{\mu(\theta)}\frac{\partial F_e}{\partial\xi}-\mu(\theta)\epsilon\frac{\partial F_e}{\partial\pi}=0, \qquad (27)$$

$$\frac{\partial\epsilon}{\partial\xi}=\int(F_i-F_e)d\pi,$$

where

$$\mu(\theta)=(1+\Omega\theta/2)^3.$$

V. NUMERICAL RESULTS (PLANAR GEOMETRY)

Numerical results have been obtained both through system (21) and system (27). The numerical integration has been performed with the help of a standard Eulerian code, slightly modified in order to take into account the time-dependent coefficients appearing in the Vlasov equations. The system (27) has turned out to be more advantageous, since it provides automatically (as we have seen) the most suitable value of the time step. With this method, a save of about 50% in computing time has been achieved.

After solving numerically the system (27) in the variables ξ and π, we come back to the usual coordinates (x,v) through (9) and (24).

There are still two parameters left to our disposal; the mass ratio M and the transformation frequency Ω. Indeed, their roles are quite different: M is a physical quantity and determines the structure of the solution in the real phase space (x,v); on the contrary, Ω is a parameter of the transformation and can only affect the structure of the (ξ,π) space.

The choice of Ω is important from a numerical point of view. In fact, for $\Omega=0$ there is no transformation at all, and the plasma expands to infinity; on the contrary, for $\Omega \gg 1$ the plasma may even experience a contraction in the (ξ,π) space. The optimum choice for Ω keeps the dimensions of the asymptotic state as close as possible to the dimensions of the initial state. All the points of the mesh are then exploited throughout the evolution. Empirically, it was found that Ω must be of the order of magnitude of the ions plasma frequency at the initial time. This is not surprising, since the rapidity with which the asymptotic solution is reached depends on the mobility of the ions.

Besides, the mobility of the ions is determined by the mass ratio M. For realistic values of M (equal, for example, to the ratio between the mass of the proton and that of the electron) the movement of the ions is extremely slow. Since we are chiefly interested in the asymptotic solution, we have taken much smaller values for M, typically going from $M=2$ to $M=10$.

In the following paragraphs, we present the results of a typical simulation for which we have chosen

$$\Omega=0.35; \quad M=4.$$

(Note that we show only the part of the phase space with $\xi,x>0$. For $\xi,x<0$, the figures are symmetric.) Space is measured in units of the electron Debye length, time in units of the electron plasma period (the inverse of the plasma frequency), and velocity in units of the electron thermal velocity $v_{\mathrm{th},e}=\lambda_{\mathrm{D}e}\omega_{p,e}$. The initial condition is uniform in x, over $12\lambda_{\mathrm{D}e}$ from $x=-6$ to $x=6$, both for ions and electrons (the plasma is then locally neutral at $t=0$). In the velocity space the distribution function is Maxwellian:

$$\exp(-m_{i,e}v^2/2T_{i,e}),$$

where T_i and T_e are, respectively, the ions and electrons temperatures. In our simulation, we have taken

$$T_e=0.7; \quad T_i=0.3.$$

In Fig. 1 we show the electron distribution function in the transformed phase space (ξ,π). The support of $F(\xi,\pi)$ experiences a slight contraction in ξ, and then freezes when the asymptotic state is reached. We can see that F develops a finer and finer structure in ξ, which ultimately leads to a loss of information on a local scale. This is a fundamental point: The rescaling transformations, confining the plasma on a finite interval, allow us to preserve the global information; nevertheless, and this is the price to pay, local information is progressively lost. In other words, the distribution function is smoothed over intervals of growing size $\Delta x(t)=C(t)\Delta\xi$.

Despite these considerations, we have to stress that (a) we are interested in the global properties of the plasma. (b) We can reasonably assume that the information that has reached the scale $\Delta x(t_0)$ at a certain time t_0 (and is therefore lost in our scheme), will not influence the phenomena at a scale larger than $\Delta x(t_0)$ for times greater than t_0. (c) Our results are supported by comparison with a simulation performed with an exact N body code. Point

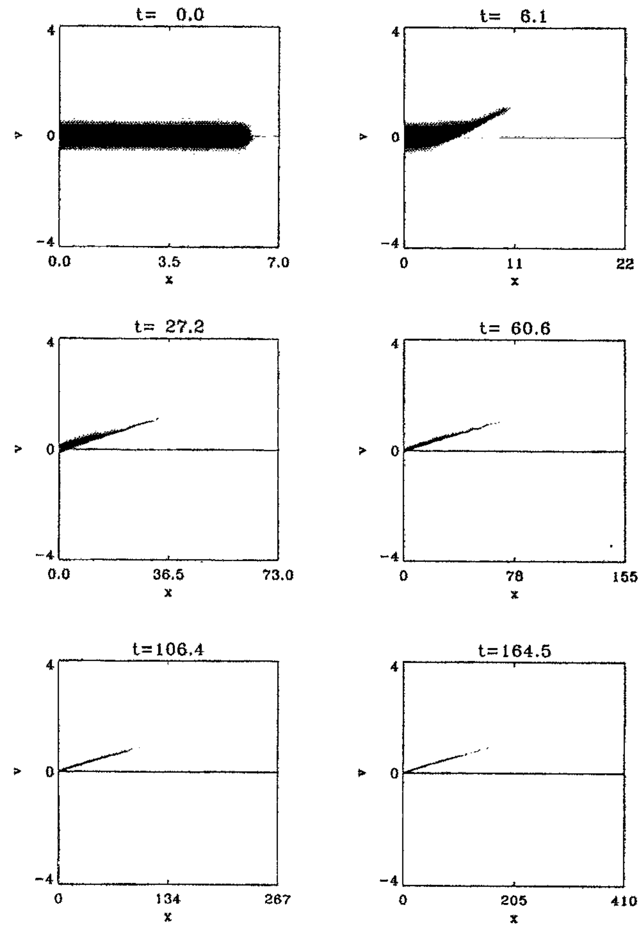

FIG. 3. Ion distribution function in the real phase space (planar geometry).

(b) has been numerically verified in Ref. 24 for an ordinary (nonrescaled) Vlasov–Poisson system.

Figures 2 and 3 show the electrons and ions distribution functions in the (x,v) phase space (note that an increasing scale is used in the x coordinate). For short times $(t=6.1)$, a layer of electrons runs ahead the bulk of the plasma, destroying the initial neutrality. The unbalance of electric charge (see Fig. 4 at the same time) generates an ambipolar electric field that accelerates the ions to high velocities. Eventually, the fast electrons are reabsorbed by the net positive charge left behind them. For large values of t, both distributions tend to assume the form

$$f(x,v,t)=n(x,t)\delta(v-x/t), \tag{28}$$

represented in the phase space by a straight line passing through the origin.

The phase portrait of Figs. 2 and 3 clearly shows that the asymptotic solution is self-similar. The space and velocity variables are strongly correlated by the relation $v=x/t$, indicating ballistic motion, in accordance with our conjecture.

In Fig. 4 we have shown the evolution of the net charge density in the rescaled space:

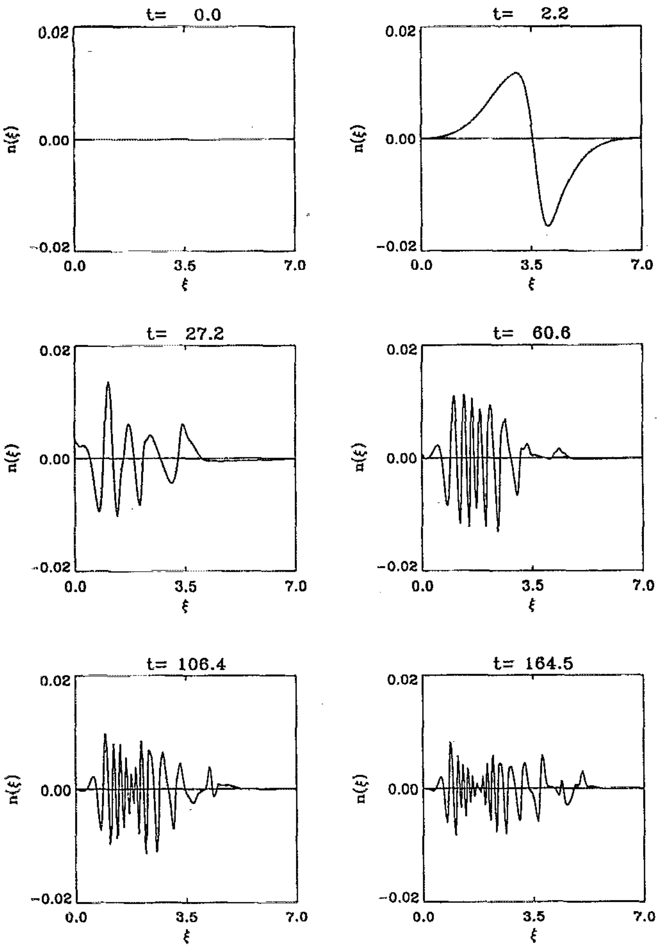

FIG. 4. Net charge density $n=n_i-n_e=\int(F_i-F_e)d\eta$ in the rescaled space (planar geometry).

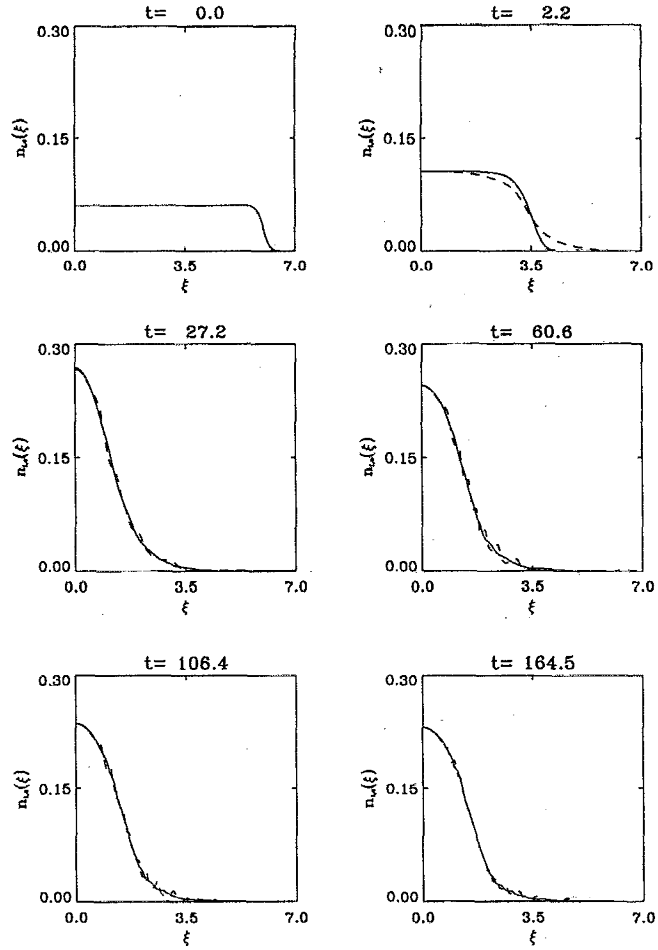

FIG. 5. Number density for the ions (solid line) and the electrons (broken line) in the rescaled space (planar geometry).

$$n(\xi)=\int[F_i(\xi,\pi)-F_e(\xi,\pi)]d\pi=n_i(\xi)-n_e(\xi).$$

We observe that charge neutrality is approached through the formation of alternatively positive and negative layers, the dimensions of which tend to zero with time. As a matter of fact, one still has to verify that the dimensions of the layers tend to zero faster than t in the rescaled space. Otherwise, their size would grow in the real space. This point will be checked via a semianalytical argument at the end of this section.

Figure 5 shows the ion density $n_i(\xi)$ (solid line) and electron density $n_e(\xi)$ (broken line) in the rescaled space. For large values of the time, they tend to assume the same profile; the decay is approximately exponential.

The time evolution of the kinetic and potential energies is plotted in Fig. 6. The kinetic energies of the two species are asymptotically constant, and proportional to the respective masses. Consequently, the electrons and the ions move asymptotically with the same velocity. The potential energy is zero at $t=0$ (we have prepared a neutral initial condition), then reaches a maximum and vanishes again for $t\rightarrow\infty$.

It might be interesting to investigate some of the thermodynamical properties and equations of state of the plasma expansion. This is an important point in order to understand the possibility of using the hydrodynamical model in collisionless plasmas. Let us define the local temperature $T(x,t)$, pressure $P(x,t)$, and density $n(x,t)$ by the

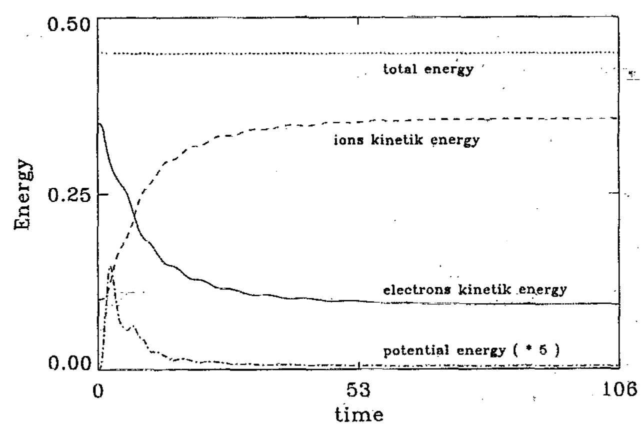

FIG. 6. Evolution of the kinetic and potential energy with time (planar geometry).

following relations:

$$kT(x,t)=m\frac{\int(v-\langle v\rangle)^2 f(x,v,t)dv}{\int f(x,v,t)dv},$$

$$P(x,t)=m\int(v-\langle v\rangle)^2 f(x,v,t)dv, \tag{29}$$

$$n(x,t)=\int f(x,v,t)dv,$$

where k is the Boltzmann constant.

Referring to the rescaling of Sec. III [see Eq. (20)], it is a matter of straightforward algebra to show that

$$T(x,t)=[1/(1+\Omega t)^2]\hat{T}(\xi,\theta),$$

$$P(x,t)=[1/(1+\Omega t)^3]\hat{P}(\xi,\theta), \tag{30}$$

$$n(x,t)=[1/(1+\Omega t)]\hat{n}(\xi,\theta),$$

where the variables with an overcaret are calculated in the rescaled space, namely

$$k\hat{T}(\xi,\theta)=m\frac{\int(\eta-\langle\eta\rangle)^2 F(\xi,\eta,\theta)d\eta}{\int F(\xi,\eta,\theta)d\eta},$$

$$\hat{P}(\xi,\theta)=m\int(\eta-\langle\eta\rangle)^2 F(\xi,\eta,\theta)d\eta, \tag{31}$$

$$\hat{n}(\xi,\theta)=\int F(\xi,\eta,\theta)d\eta.$$

In Fig. 7 we have plotted the graph of the ion pressure $\hat{P}(\xi,\theta)$ for large values of the time, when the asymptotic solution has approximately been reached. It appears that eventually $\hat{P}$ does not depend on θ anymore, neither do $\hat{n}$ (see Fig. 5) and $\hat{T}$, which is the ratio between the pressure and the density.

A first result arising from Eq. (31) is that the local temperature decreases in time as t^{-2}. This is a crucial point, since it invalidates those hydrodynamic models that are based on the assumption of isothermality. During the expansion, the initial thermal energy of the plasma is progressively transformed into drift energy. Asymptotically the plasma is cold and all its energy derives from the drift, ballistic velocity, while the potential energy goes to zero.

As a matter of fact, the relation (31) is even more profound, and can be used to check the thermodynamics of the system. Let us suppose that the plasma obeys a polytropic equation of state:

$$\frac{d}{dt}(Pn^{-\gamma})=\left(\frac{\partial}{\partial t}+\langle v\rangle\frac{\partial}{\partial x}\right)(Pn^{-\gamma})=0, \tag{32}$$

where γ is the polytropic coefficient.

If one chooses $\gamma=3$, the ratio P/n^3 becomes

$$(Pn^{-3})(x,t)=(\hat{P}\hat{n}^{-3})(\xi),$$

which is not dependent on θ.

Since, asymptotically, $\xi=x/(\Omega t)$, the ratio P/n^3 becomes a function of x/t only, and its total derivative is therefore equal to zero. We have, in fact,

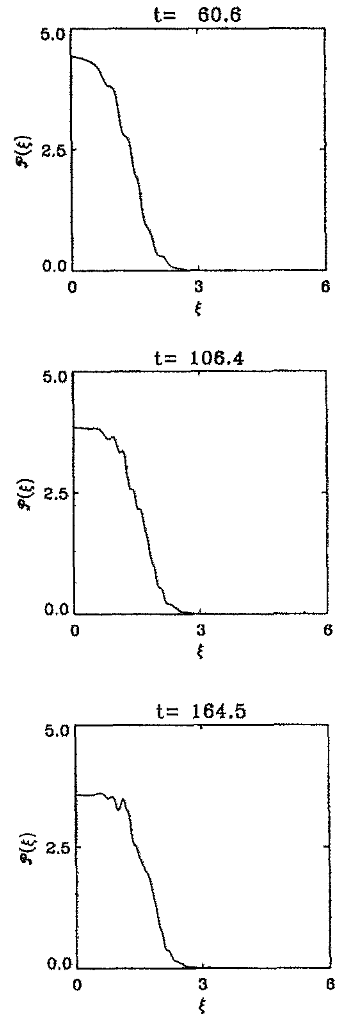

FIG. 7. Ion pressure $\hat{P}=\int(\eta-\langle\eta\rangle)^2F(\xi,\eta,\theta)d\eta$ in the rescaled space (planar geometry).

$$\langle v\rangle=\frac{x}{t};\quad \frac{\partial}{\partial t}=-\frac{x}{\Omega t^2}\frac{\partial}{\partial\xi};\quad \frac{\partial}{\partial x}=\frac{1}{\Omega t}\frac{\partial}{\partial\xi}, \tag{33}$$

and therefore

$$\left(\frac{\partial}{\partial t}+\langle v\rangle\frac{\partial}{\partial x}\right)[\hat{P}\hat{n}^{-3}(\xi)]=0.$$

This result strongly claims that the only reasonable value of the polytropic coefficient (at least for this one-dimensional problem) is $\gamma=3$. On the other hand, it shows that the polytropic relation must be imperatively written in its most general form (32), rather than in the (more usual) form,[14,15]

$$Pn^{-3}=\text{const}, \tag{34}$$

which implies that the ratio is constant both with respect to x and to t.

Moreover, we must use two pressures: one connected to the electron density and the other connected to the ion density (partial pressures of each specie). We shall therefore have two polytropic equations, and in each momentum (Euler) equation we must use only the partial pressure of each single specie. This expresses the fact that all exchanges of momentum between the two species take place only through the electric field.

The previous results have confirmed that the assumption of the hydrodynamic model are verified only in very special cases. It turns out that, when the evolution of the plasma is self-similar (as it is our case), such an assumption holds, and the hydrodynamic model is quite accurate, provided that γ is chosen equal to 3 (for one-dimensional systems). From this point of view, the existence of self-similar solutions is not a consequence of the hydrodynamics equations, but rather a condition for the validity of the model.

We are now in a position to check whether the size of the charged layers observed in Fig. 4 goes to zero with time also in the real space. Let us calculate the Debye length:

$$\lambda_D=(\epsilon_0 kT/ne^2)^{1/2}.$$

Since we have shown that the temperature varies asymptotically as t^{-2} and the density as t^{-1}, it turns out that $\lambda_D \sim t^{-1/2}$. The Debye length then goes to zero, even in the real space, indicating that, asymptotically, no region of non-neutrality can subsist on a finite scale.

VI. EXPANSION IN SPHERICAL GEOMETRY

In spherical coordinates, the Vlasov equation reads (details can be found in Ref. 25) as

$$\frac{\partial f}{\partial t}+R\frac{\partial f}{\partial r}+\frac{\Theta}{r}\frac{\partial f}{\partial \theta}+\frac{\phi}{r\sin\theta}\frac{\partial f}{\partial \psi}+\left(\frac{\Theta^2+\phi^2}{r}+E_r\right)\frac{\partial f}{\partial R}$$
$$+\left(\frac{\phi^2}{r}\frac{1}{tg\theta}-\frac{R\Theta}{r}+E_\theta\right)\frac{\partial f}{\partial \Theta}$$
$$-\left(\frac{R\phi}{r}+\frac{\Theta\phi}{r}\frac{1}{tg\theta}-E_\psi\right)\frac{\partial f}{\partial \phi}=0, \tag{35}$$

where r, θ, ψ are the spatial coordinates, and R, Θ, ϕ the respective components of the velocity.

If we assume radial symmetry,

$$\frac{\partial f}{\partial \theta}=\frac{\partial f}{\partial \psi}=E_\theta=E_\psi=0,$$

then f depends only on t, r, R, and $T=\Theta^2+\phi^2$, and we can write

$$f(t,r,R,\Theta,\phi)=(1/\pi)\tilde{f}(t,r,R,T)$$

(the factor π is introduced for notation convenience). A little algebra shows that, in the case of radial symmetry, Eq. (35) becomes

$$\frac{\partial \tilde{f}}{\partial t}+R\frac{\partial \tilde{f}}{\partial r}+\left(\frac{T}{r}+E_r\right)\frac{\partial \tilde{f}}{\partial R}-\frac{2RT}{r}\frac{\partial \tilde{f}}{\partial T}=0. \tag{36}$$

A further simplification consists in taking the tangential velocities T equal to zero. This is done by imposing the following condition:

$$\tilde{f}(t,r,R,T)=g(t,r,R)\cdot\delta(T), \tag{37}$$

where δ is the Dirac delta function.

Integrating (36) with respect to T, we obtain

$$\frac{\partial g}{\partial t}+R\frac{\partial g}{\partial r}+E_r\frac{\partial g}{\partial R}+\frac{2R}{r}g=0. \tag{38}$$

Finally, we perform the transformation

$$\varphi(t,r,R)=r^2 g(t,r,R).$$

Note that $gr^2\,dr\,dR=\varphi\,dr\,dR$ represents the normalized number of particles contained in a shell of thickness dr, with radial velocity ranging from R to $R+dR$: it is, in fact, this quantity that is conserved, and not simply $g\,dr\,dR$. The resulting Vlasov–Poisson system for a two-component plasma then reads as

$$\frac{\partial \varphi_i}{\partial t}+R\frac{\partial \varphi_i}{\partial r}+\frac{E_r}{M}\frac{\partial \varphi_i}{\partial R}=0,$$
$$\frac{\partial \varphi_e}{\partial t}+R\frac{\partial \varphi_e}{\partial r}-E_r\frac{\partial \varphi_e}{\partial R}=0, \tag{39}$$
$$\frac{\partial}{\partial r}(r^2E_r)=\int(\varphi_i-\varphi_e)dR.$$

The first and second equations of (39) are identical to the one-dimensional planar Vlasov equation, with the usual normalization:

$$\int \varphi_{i,e}\,dr\,dR=1.$$

The approximation (37) allows us to work in a bidimensional phase space (r,R) by neglecting all tangential velocities. The price we have to pay for this simplification is that the point $r=0$ becomes singular: in fact, nothing prevents particles with negative radial velocities (inward bound) to arrive at the origin, and consequently generating an infinite electric field. This will restrict the number of initial conditions that can be treated by our model.

Rescaling. We use the "variable mass" rescaling technique, as we have done in the plane geometry problem. The transformation relations are, in the spherical case,

$$r=C(t)\xi,$$
$$dt=A^2(t)d\theta,$$
$$mR=\pi/C+m\dot{C}\xi, \tag{40}$$
$$\varphi_{i,e}(r,R,t)=F_{i,e}(\xi,\pi,\theta),$$
$$E_r(r,t)=\epsilon(\xi,\theta).$$

In the choice of the expansion factor $C(t)$, we are guided by the results obtained in the planar problem. The only difference lies in the form of the Poisson equation, which expresses the fact that the electric field now vanishes as r^{-2} for $r\to\infty$. Consequently, we can again expect a ballistic

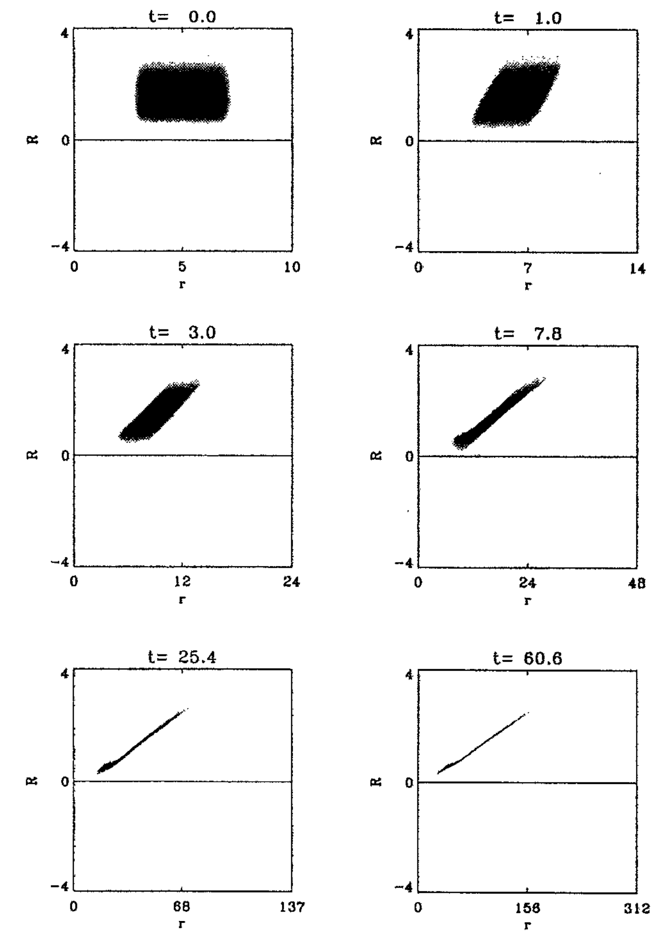

FIG. 8. Electron distribution function in the real phase space (spherical geometry).

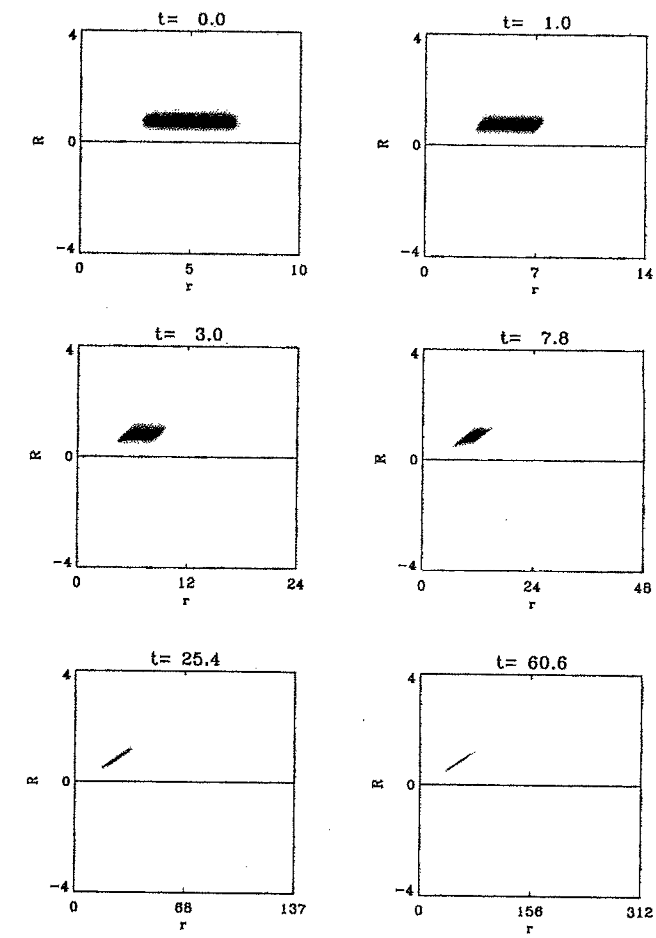

FIG. 9. Ion distribution function in the real phase space (spherical geometry).

asymptotic expansion, although not necessarily local neutralization. Let us make therefore the following choice:

$$C(t)=1+\Omega t.$$

Moreover, we want to choose $A(t)$, so that

$$A^2(t)=\omega_p^{-1}(t).$$

Now the density $n(r)=\int \varphi dR$ decreases as $1/r^3$; thus we have

$$\omega_p \propto \sqrt{n} \propto r^{-3/2} \propto C^{-3/2},$$

and finally

$$A(t)=C^{3/4}(t)=(1+\Omega t)^{3/4}.$$

Integrating the second of (40), we obtain

$$1+\Omega t=(1-\Omega\theta/2)^{-2},$$

the new time θ being now renormalized between $0 \leqslant \theta \leqslant 2/\Omega$.

With these choices, the system (39) is transformed into the following one:

$$\frac{\partial F_i}{\partial \theta}+\frac{\pi}{M\mu(\theta)}\frac{\partial F_i}{\partial \xi}+\mu(\theta)\epsilon\frac{\partial F_i}{\partial \pi}=0,$$

$$\frac{\partial F_e}{\partial \theta}+\frac{\pi}{\mu(\theta)}\frac{\partial F_e}{\partial \xi}-\mu(\theta)\epsilon\frac{\partial F_e}{\partial \pi}=0, \tag{41}$$

$$\frac{\partial}{\partial \xi}(\xi^2\epsilon)=\int (F_i-F_e)d\pi,$$

where

$$\mu(\theta)=(1-\Omega\theta/2)^{-1}.$$

VII. NUMERICAL RESULTS (SPHERICAL GEOMETRY)

We have solved numerically the system (41). The same considerations on the roles of the parameters M and Ω can be done, as in the planar geometry problem. In the following, we shall present a simulation for which we have chosen $M=4$ and $\Omega=0.5$.

In order to prevent particles from arriving at the origin, we have prepared an initial condition where all the particles are situated outside a sphere of radius r_0 and have positive (outward bound) velocities. In particular, the ini-

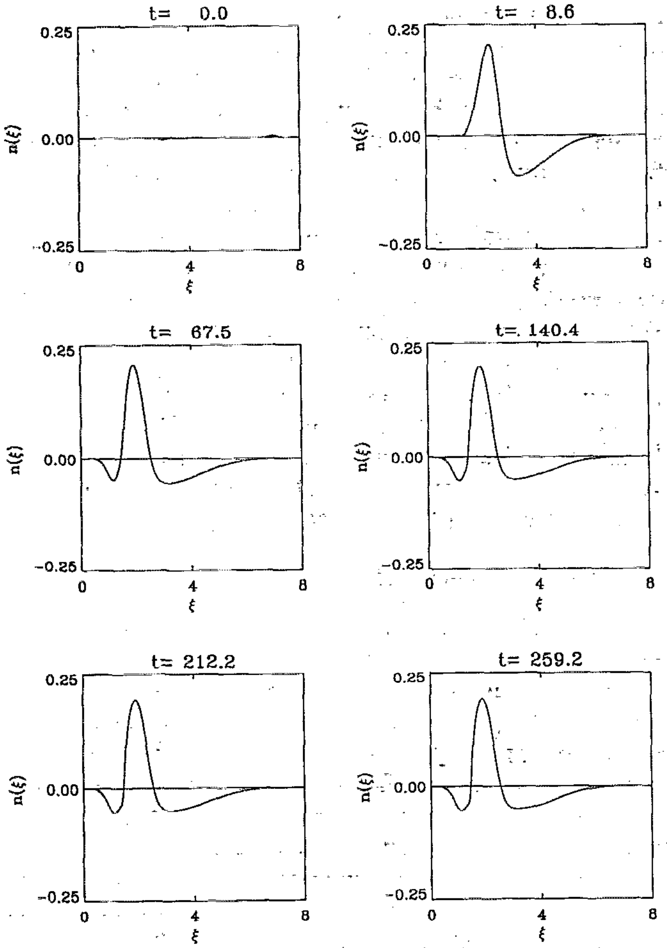

FIG. 10. Net charge density in the rescaled space (spherical geometry).

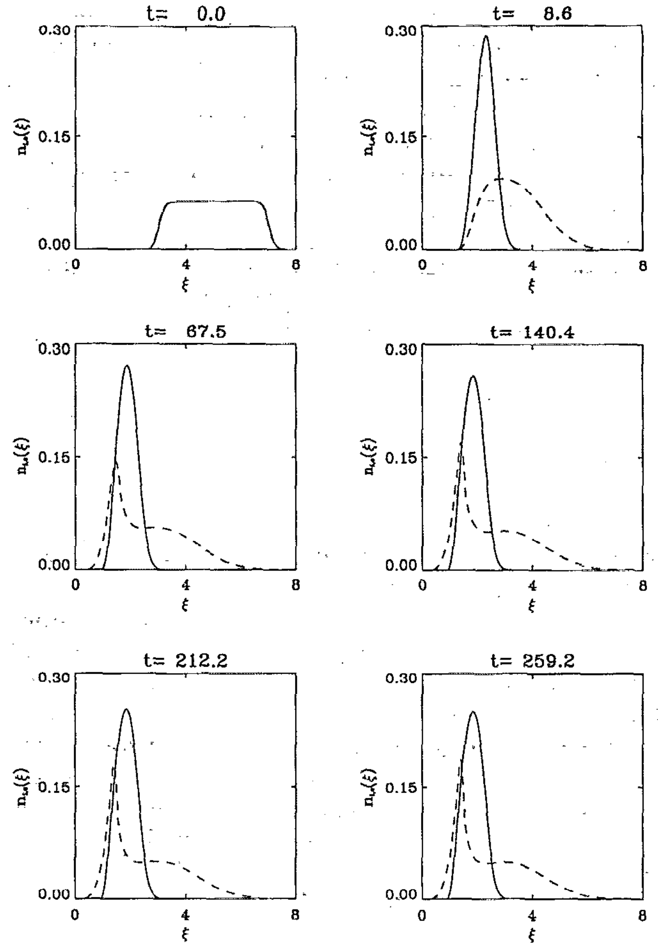

FIG. 11. Number density for the ions (solid line) and the electrons (broken line) in the rescaled space (spherical geometry).

tial distribution function is uniform in r between $r=3$ and $r=7$, and zero elsewhere. In the velocity space, it has the following profile:

$$(R-R_0)^2 \exp[-m_{i,e}(R-R_0)^2/2T_{i,e}], \qquad R \geqslant 0,$$
$$0, \qquad R<0,$$

where $R_0=0.4$, $T_i=0.2$, and $T_e=0.6$.

The phase portrait for electrons and ions is represented in Figs. 8 and 9, respectively. As in the planar case, a strong correlation is found between the space and velocity variable ($R=r/t$). The solution is asymptotically self-similar and again corresponds to ballistic motion.

Figure 10 shows the evolution of the net charge density in the rescaled space, and Fig. 11 shows the number densities of electrons and ions. A multiple layer, non-neutral structure is soon created, which seems to be very stable over long times. The phenomenon of local neutralization found in planar geometry no longer takes place. The evolution of the root mean square of the radius shows that the motion of the ions and the electrons is completely decoupled (Fig. 12). The two species move asymptotically with constant but different velocities.

From the previous results, it is clear that the asymptotic behavior of the plasma expansion into vacuum strongly depends on the dimensionality of the system. The essential point, differentiating the planar problem from the spherical one, lies in the form of the Poisson equation. In a planar one-dimensional geometry, the electric field generated by one "particle" (in fact, a charged sheet) is uniform, and does not vanish for $x\to\pm\infty$. Such a field does not admit an "escape velocity," and therefore even the fastest electrons are eventually reabsorbed by the plasma.

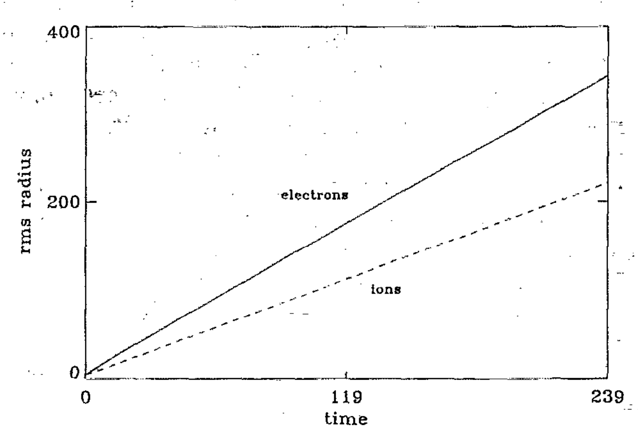

FIG. 12. Evolution of the root mean square of the radius for the ions (broken line) and the electrons (solid line) (spherical geometry).

On the other hand, in the one-dimensional spherical geometry, the field created by a charged sphere falls down as r^{-2}: consequently, an escape velocity exists, just as in a gravitational system, and a cloud of fast electrons is able to escape from the bulk of the plasma.

The stability of the non-neutral structure seen in Fig. 10 can be justified by means of an easy calculation in the transformed space.

From a Lagrangian point of view, the equations of motion corresponding to system (41) are the following (again for electrons of unitary mass):

$$\frac{d\xi}{d\theta}=\frac{\pi}{\mu(\theta)}\equiv\eta,$$
$$\frac{d\pi}{d\theta}=\mu(\theta)\epsilon(\xi,\theta), \tag{42}$$

where $\mu(\theta)=(1-\theta)^{-1}$. We have taken $\Omega=2$ for convenience of notation; θ is then normalized on $[0,1[$.

Let us suppose that, at the time $\theta=\theta_0$, the multiple layer structure has already formed, giving rise to a field $\epsilon(\xi,\theta_0)$, which is finite for every ξ. From the last of Eqs. (40), we know that $\epsilon(\xi,\theta_0)=E(x,t_0)$, t_0 being the time corresponding to θ_0. There is no physical reason for the real field E to diverge for $t>t_0$; thus ϵ will also remain finite for $\theta>\theta_0$. On this basis, we can assume that the variations if ϵ with θ are small compared to the variations of $\mu(\theta)$.

Integrating the second of (42), we obtain

$$\pi(\theta)=\epsilon\int_{\theta_0}^{\theta}\mu(\theta)d\theta=-\epsilon\log\frac{1-\theta}{1-\theta_0}. \tag{43}$$

Hence, π diverges for $\theta\rightarrow 1$.

The displacement $\Delta\xi$ between θ_0 and 1 is

$$\Delta\xi=-\epsilon\int_{\theta_0}^{1}(1-\theta)\log\frac{1-\theta}{1-\theta_0}d\theta. \tag{44}$$

Evaluating the integral in Eq. (44) gives

$$\Delta\xi=\epsilon[(1-\theta_0)/4]. \tag{45}$$

Equation (45) shows that the displacement tends to zero for $\theta_0\rightarrow 1$. In other words, even if between $\theta=\theta_0\approx 1$ and $\theta=1$ an infinite time has elapsed, the particles have been moving on a negligible distance $\Delta\xi$. The non-neutral structure seen in Figs. (10) and (11) is then completely stable over arbitrarily long times. Moreover, this result confirms *a posteriori* our previous assumption that $\epsilon(\theta)$ varies slowly with respect to $\mu(\theta)$: the proof is then self-coherent.

One could be tempted to interpret such a structure as an indication of a Debye length growing as t in the real space. In fact, this is not the case, as it can be shown by a calculation similar to the one performed at the end of Sec. V.

Now the density decreases as t^{-3}, whereas the temperature varies as t^{-2}. Consequently, the Debye length should vary as $t^{1/2}$ in the real space, and as $t^{-1/2}$ in the rescaled space. This result is apparently not in agreement with what we observe in Fig. 10, where the region of non-neutrality is clearly fixed in the rescaled space, and therefore grows as t in the real space. In fact, there is no paradox, for the separation distance observed in Fig. 10 is *not* a Debye length. The separation of the two species is due to the fact that, because of the radial symmetry, particles situated far away from the origin virtually behave as free particles, their Coulomb interaction being negligible. Had the two species been two neutral gases, we would have observed a similar decoupling, which is essentially due to the different initial conditions, and by no means to electrostatic phenomena. In this sense, we cannot speak of a Debye length.

On the other hand, electrostatic effects could be important in the central region of the plasma, which we have ignored in order to keep a bidimensional phase space. A further investigation, using a 3-D phase space code, should verify if, in the central region, the Debye length actually behaves as predicted by the above calculations.

VIII. CONCLUSIONS AND OPEN PROBLEMS

The results obtained in this paper present a double interest.

From a mathematical and computational point of view, we have given an example of how the rescaling methods can be applied to an expansion problem. The interest of these methods lies in the combination of analytical and numerical tools. In a few special cases semianalytical solutions can be easily obtained (see, for example, Ref. 19). More generally, the rescaling suggests reasonable conjectures on the structure of the time asymptotic solution and provides an intelligent, easily implementable numerical scheme. In our case, the time asymptotic solution is the ballistic expansion, and it is automatically (i.e., analytically) taken into account by the rescaling transformation.

From a physical point of view, we have investigated the long-standing problem of plasma expansion into vacuum. Most previous works on this subject, which have been summarized in the Introduction, are based on the hydrodynamic model, and lead, in some special cases, to a set of self-similar solutions. However, their numerical solution shows an anomalous behavior (collapse of the ion front) after a relatively brief time, indicating that the model is not accurate to treat this problem.

In this work, the plasma expansion was treated through a kinetic, collisionless model (the Vlasov–Poisson system). In the case of a one-dimensional planar expansion, we found numerically that, for large values of the time, the solution is indeed self-similar and characterized by the relation $v=x/t$. The plasma becomes locally neutral and consequently exhibits ballistic motion.

One of the aims of this paper was to check the domain in which the hydrodynamic model is suited to treat a plasma expansion into vacuum. In particular, one has to verify whether the polytropic relation $(d/dt)(pn^{-\gamma})=0$ holds, and which value of γ has to be used. The numerical results suggest that such relation is not satisfied during the first instants of the expansion, when the asymptotic solution has not yet been reached. In this case, the kinetic treatment is necessary to take into account all the details of the phase space distribution function. Nevertheless, we showed that, over long times, when the self-similar solution is well established, the poytropic relation holds within a good approximation. Our analytical and numerical calculations clearly showed that the polytropic coefficient must assume the value $\gamma=3$ for a one-dimensional planar expansion. The previous result is, in fact, quite general and proves that the polytropic relation is verified for every self-similar expansion.

Another important result arose from the study of the expansion in a spherical, one-dimensional geometry. Once again, the solution over long times is self-similar (ballistic motion). Yet, the process of local neutralization found in the planar geometry case does not take place; contrarily, we observed the formation of charged layers of finite dimension with respect to the total length of the system. The different behavior in the planar and spherical cases was interpreted in terms of the different form of the Poisson equation in the two geometries. Finally, we proved that such multiple layer structure is stable over long times.

A further insight into the spherical expansion could be achieved by taking into account tangential velocities T, and then working in a three-dimensional phase space (r,R,T). More general initial conditions could thus be treated, but the requested numerical effort would be considerably stronger.

The analog quantum-mechanical problem (Schrödinger–Poisson system) is also of interest: an open question is whether quantum effects can corrupt the classical solution. Both problems are, at present, under study.

ACKNOWLEDGMENTS

The authors wish to thank Dr. E. Fijalkow for helpful assistance in the use of supercomputing machines.

This work was supported by the Associazione Sviluppo Scientifico e Tecnologico del Piemonte, Torino, Italy.

[1] R. Tanberg, Phys. Rev. **35**, 1080 (1930).
[2] E. Kobel, Phys. Rev. **36**, 1636 (1930).
[3] A. A. Plyutto, Sov. Phys. JETP **12**, 1106 (1961).
[4] A. V. Gurevich, L. P. Pitoevskii, and V. V. Smirnova, Space Sci. Rev. **9**, 805 (1969).
[5] N. Singh and R. W. Schunk, J. Geophys. Res. **93**, 12901 (1988).
[6] P. Wagli and T. P. Donaldson, Phys. Rev. Lett. **40**, 875 (1978).
[7] J. S. Pearlman, J. J. Thomson, and C. E. Max, Phys. Rev. Lett. **38**, 1397 (1977).
[8] V. G. Eselevich and V. G. Fainshtein, Sov. Phys. JETP **52**, 441 (1980).
[9] C. Chan, N. Hershkowitz, A. Ferreira, T. Intrator, B. Nelson, and K. Longren, Phys. Fluids **27**, 226 (1984).
[10] G. Hairapetian and R. L. Stenzel, Phys. Fluids B **3**, 899 (1991).
[11] A. V. Gurevich, L. V. Pariiskaya, and L. P. Pitaevskii, Sov. Phys. JETP **22**, 449 (1966).
[12] J. E. Allen and J. Andrews, J. Plasma Phys. **4**, 187 (1970).
[13] J. E. Crow, P. L. Auer, and J. E. Allen, J. Plasma Phys. **14**, 65 (1975).
[14] Ch. Sack and H. Schamel, Phys. Rep. **156**, 311 (1987).
[15] Ch. Sack and H. Schamel, Plasma Phys. Controlled Fusion **27**, 717 (1985).
[16] J. Denavit, Phys. Fluids **22**, 1384 (1979).
[17] M. Galvez and J. E. Borovsky, Phys. Fluids B **3**, 1892 (1991).
[18] H. Zrineh, M. R. Feix, E. Fijalkow, and M. Navet, Trans. Theory Stat. Phys. **16**, 279 (1987).
[19] J. R. Burgan, M. R. Feix, E. Fijalkow, and A. Munier, J. Plasma Phys. **29**, 139 (1983).
[20] J. R. Burgan, M. R. Feix, E. Fijalkow, and A. Munier, Phys. Lett. A **74**, 11 (1979).
[21] J. R. Burgan, J. Gutierrez, A. Munier, E. Fijalkow, and M. R. Feix, in *Strongly Coupled Plasmas*, edited by G. Kalman (Plenum, New York, 1978), p. 597.
[22] J. L. Duranceau, S. Bouquet, E. Fijalkow, and M. R. Feix, Phys. Lett. A **98**, 309 (1983).
[23] G. W. Bluman and J. D. Cole, *Similarity Methods for Differential Equations* (Springer-Verlag, Berlin, 1974).
[24] P. Bertrand (private communication, 1991).
[25] A. Munier, M. R. Feix, E. Fijalkow, J. R. Burgan, and J. Gutierrez, Astron. Astrophys. **78**, 65 (1979).