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On some analogies concerning the *N*-body problem, quantum billiards and the refraction of a light beam

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Abstract. Several analogies between simple physical systems are presented. A quantum mechanical argument is used to give direct proof of the equivalence between the one-dimensional N body problem (with hard particles) and the problem of a single particle bouncing in a billiard delimited by an N-dimensional simplex. Then, for a system made of two soft particles, we prove the equivalence with an optical reflection/refraction problem.

Résumé. On présente ici quelques analogies reliant des systèmes physiques simples. A l'aide d'un argument quantique, on montre l'équivalence entre le problème de N masses impénétrables à une dimension, et celui d'une seule particule dans un billiard à N dimensions en forme de simplex. Pour le probléme de deux corps pénétrables, on montre l'équivalence avec un problème d'optique de réflection et réfraction.

1. Introduction

The problem of N hard particles of different masses, moving on a line and interacting through elastic collisions has received much attention during the last 20 years. In fact, this simple system seems very promising to provide new insights into the foundation of statistical mechanics.

Furthermore, computer experiments can be carried out in a straightforward and very precise way, since the equation of motion can be integrated exactly, obviously up to the machine precision for floating real numbers. Many works have focused on the ergodic properties of such a system: some of the most recent numerical results (Rouet *et al* 1993) indicate that the system is always ergodic for $N > 2$ (except for the trivial case of all equal masses). When $N = 2$, the system is probably also ergodic, except for a countable set of mass ratios; also, it has been shown, for $N > 2$, that some particular initial conditions give rise to periodic orbits for any mass ratio (Rabouw and Ruijgrok 1981).

On the other hand, the literature on classical billiards is extremely large, and we shall not try the impossible task of making an even partial bibliography on this subject. By definition, a billiard is a system composed of one point particle moving inside an N-dimensional volume, and bouncing elastically against its boundary. Depending on the shape of the boundary, the orbits can be periodic or chaotic.

The fundamental point, which is common to both the one-dimensional N-body and the billiard system, is that velocity space and configuration space are totally decoupled and can each be handled separately. In fact, the velocities of two particles after they have collided depend only on their velocities before the collision, and not on their positions. This remarkable property is due to the fact that, in both systems, there is no potential, but only kinetic energy. The interactions between the particles (or between one particle and the boundary) are expressed through a geometric law (perfect reflection) rather than a physical one (which would imply the existence of a potential, and thus of a potential energy).

In fact, the analogy between these two systems can be brought further, as it has been shown in a number of papers. The aim of the article is to present such a parallel in a far simpler way, as well as to add a few original contributions. In section 2 we state the fundamental property that links the N-body and the billiard systems, and prove it by means of quantum mechanical arguments. Incidentally, this will extend the demonstration to the quantum domain. In section 3, we present another analogy, now involving

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particles with soft cores (i.e., penetrable particles). In section 4 we conclude.

2. Billiards and N-body problems

Hobson (1975) proved the following very remarkable theorem: the problem of two hard particles of masses m_1 and m_2 bouncing elastically in a one-dimensional box delimited by two hard walls is strictly equivalent to that of a particle moving inside a right triangle having an interior angle of $\tan^{-1}(m_2/m_1)^{1/2}$ ('triangular billiard'). This result was generalized by Foidl and Kasperkovitz (1988) to the N-body problem, which was proved to be equivalent to a billiard in a N-dimensional simplex, defined as follows:

$$\{(x_1, x_2, \ldots x_N) \in \mathbb{R}^N:\ 0 < x_1/\sqrt{m_1} < x_2/\sqrt{m_2} < \ldots < x_N/\sqrt{m_N} < 1\}.$$

However, the demonstration given in Foidl and Kasperkovitz (1988), is based on quite complicated algebraic arguments. In this paper, we shall show that, by considering the same problem from the viewpoint of quantum mechanics, the above proof turns out to be straightforward, and easily generalized to any number of particles. Then, nothing prevents us from applying Bohr's correspondence principle to argue that the demonstration must also be valid in the classical limit (i.e. when $\hbar \to 0$).

In order to fix the ideas, let us consider $N = 2$ particles of masses m_1 and m_2 in a one-dimensional box of unit length. The walls are situated at $q = 0$ and $q = 1$. The particles interact elastically between themselves and with the walls. Let us call q_1 and q_2 respectively the position of the first and second particle. Since the particles cannot cross each other, the only accessible zone of the configuration space is the surface (in fact, a right isosceles triangle) such that: $0 \leq q_1 \leq q_2 < 1$ (see figure 1).

In quantum mechanics such a system is described by a wavefunction $\psi(q_1, q_2, t)$, which obeys the Schrödinger equation:

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m_1}\frac{\partial^2\psi}{\partial q_1^2} - \frac{\hbar^2}{2m_2}\frac{\partial^2\psi}{\partial q_2^2} + V(q_1, q_2)\psi$$

where $V(q_1, q_2) = V_0[\delta(q_1 - q_2) + \delta(q_1) + \delta(q_2 - 1)]$, $V_0 \to \infty$. In this case, the potential being infinite, if the wavefunction is initially zero in the region outside the isosceles triangle of figure 1, it will remain equal to zero for any time, and the problem reduces to the free-particle Schrödinger equation:

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m_1}\frac{\partial^2\psi}{\partial q_1^2} - \frac{\hbar^2}{2m_2}\frac{\partial^2\psi}{\partial q_2^2} \tag{1}$$

endowed with the following boundary conditions:

$$\psi(q_1 = 0, q_2, t) = \psi(q_1, q_2 = 1, t) = \psi(q_1 = q_2, t) = 0$$

meaning that the wavefunction is zero outside the triangle.

We now perform the following rescaling:

$$x = \sqrt{m_1}q_1 \qquad y = \sqrt{m_2}q_2 \tag{2}$$

which transforms the Schrödinger equation (1) into the following one:

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2}\left(\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2}\right) \equiv -\frac{\hbar^2}{2}\Delta_2\psi \tag{3}$$

Δ_2 being the Laplacian in two dimensions.

In the new variables, $\psi(x, y, t)$ must be zero on the boundary of the right triangle defined by the relation $0 \leq x/\sqrt{m_1} \leq y/\sqrt{m_2} \leq 1$ (figure 2). Now, the Schrödinger equation (3), with the above boundary condition, describes the motion of a particle of unit mass in the triangular billiard shown in figure 2. As is proven in Hobson (1975), the internal angle is given by $\alpha = \tan^{-1}(m_2/m_1)^{1/2}$.

The demonstration can be trivially generalized to the N-body case by defining

$$x_1 = \sqrt{m_1}q_1 \ldots x_N = \sqrt{m_N}q_N$$

and obtaining the N-dimensional Laplacian in the Schrödinger equation. The equivalent billiard will

Figure 1. The configuration space for the two-body problem is the shaded right isosceles triangle: $0 \leq q_1 \leq q_2 \leq 1$.

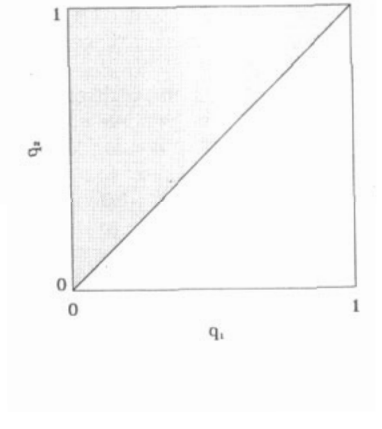
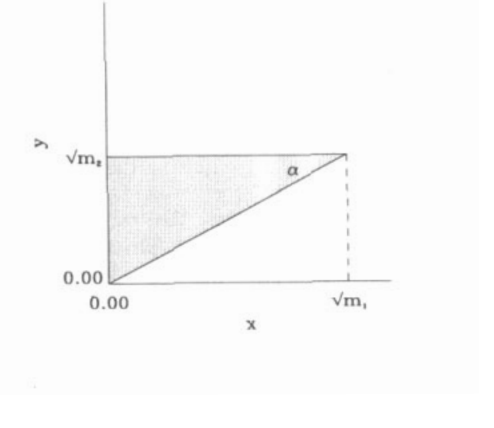


Figure 2. The equivalent billiard is the shaded right triangle, the sides of which are proportional to the square root of the masses.



therefore be given by the simplex

$$0 \le \frac{x_1}{\sqrt{m_1}} \le \frac{x_2}{\sqrt{m_2}} \le \ldots \le \frac{x_N}{\sqrt{m_N}} \le 1. \qquad (4)$$

This is the result found classically by Foidl and Kasperkowitz (1988) through much longer calculations. Our approach proves that it keeps its validity in the quantum mechanical domain. Furthermore, the demonstration only implying simple geometrical arguments, it must remain valid when the classical limit is taken.

As a matter of fact, once the result has been established, one can go back to classical physics and try to recover a similar proof.

The Hamiltonian of the two-body problem is

$$H(q_1, q_2, p_1, p_2) = \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} + V_0[\delta(q_1) + \delta(q_1 - q_2) + \delta(q_2 - 1)] \qquad (5)$$

with $V_0 \to \infty$.

Let us perform the following (canonical) transformation:

$$x = q_1\sqrt{m_1} \qquad y = q_2\sqrt{m_2} \qquad p_x = p_1/\sqrt{m_1} \qquad p_y = p_2/\sqrt{m_2} \qquad (6)$$

which gives the new Hamiltonian;

$$K(x, y, p_x, p_y) = \frac{p_x^2}{2} + \frac{p_y^2}{2} + V_0[\delta(x) + \delta(x\sqrt{m_2/m_1} - y) + \delta(y - \sqrt{m_2})] \qquad (7)$$

where we have used the property of the delta function $\delta(x/a) = a\delta(x)$, and then absorbed the constant a in the factor V_0, which tends to infinity.

The Hamiltonian (7) can be regarded as describing the motion of a unit mass particle in the triangle delimited by the straight lines;

$$x = 0; \; y = x\sqrt{m_2/m_1}; \; y = \sqrt{m_2}$$

which is plotted in figure 2. Since $V_0 \to \infty$, a particle with initial coordinates (x_0, y_0) inside the triangle will remain inside for all times, and bounce elastically against the walls.

As to the N-body problem, we follow the same procedure to get the equations of the $N+1$ straight lines that delimit the simplex (4).

In summary, it becomes now clearer why the proof is more direct in the frame of quantum mechanics. The correct rescaling for the momentum is the one that eliminates the masses from the kinetic part of the Hamiltonian; then, the rescaling of the position variable is obtained by requiring the transformation to be canonical.

In quantum mechanics the canonical relation between position and momentum is automatically satisfied by the formalism; also, the potential part of the Hamiltonian can be intuitively treated (as is usually done) by imposing the correct boundary conditions for the Schrödinger equation. Then, the proof follows as an immediate consequence.

3. Another analogy: soft-core particles and diffraction

So far, we have considered the case of hard-core particles, for which $V_0 \to \infty$, and thus the particles cannot cross each other. Indeed, this property was fundamental to prove our result: it is the relation of order among the particles that defines the boundary of the N-dimensional simplex.

The next step, which we undertake in this section, would naturally consist in considering a system of soft-core particles, for which $V_0 < \infty$, so that the particles can cross each other if their relative kinetic energy is sufficiently high.

Let us be more precise, and analyse what happens when two particles collide. First of all, we remark that two particles can interact only when they are at the same point, since the interaction potential is still of the form $V(q_1, q_2) = V_0\delta(q_1 - q_2)$ (zero-range interaction). Now, the kinetic energy of the couple can be decomposed into the energy *of* its centre of mass (hereafter called K_G) plus the energy of the particles *in* the reference of their centre of mass (referred to as K_R), with of course

$$E_T = K_G + K_R.$$

Thus, when two particles find themselves at the same point:
(a) if $K_R < V_0$, they do not cross each other, and modify their velocities just as in the hard-core case:
(b) if $K_R > V_0$, they cross each other without any change in velocity.

Remember that, for the hard-core particles, the accessible region of the configuration space was the simplex $\{0 < x_1 < x_2 < \ldots < x_N < 1\}$. Now, the relation of order is no more satisfied, and therefore all the N-dimensional cube of unit side is accessible. In fact, when two particles cross each other, the representative point of the system in configuration space jumps from one simplex to another. There are $N!$ ways of rearranging our particles, so that there must exist $N!$ different simplexes in the accessible configuration space. It is indeed a known result that the N-dimensional unit cube can be covered with $N!$ simplexes.

We now come at the main result of this section, which may be stated as follows: *the above system of $N = 2$ soft particles in one dimension is equivalent to the propagation of a beam of light in a rectangle with perfectly reflecting sides, and in which one diagonal is constituted of a thin slide of some refracting material (with refractive index $n < 1$), while the rest is vacuum* ($n = 1$). Just as in the hard–core case, the sides of the rectangle are in the same ratio as the square root of the masses of the particles; in addition, the refractive index n is a function of the potential barrier V_0, which we will determine later on.

Unfortunately, such an amusing result is no more valid for $N \geq 3$, and the reason why will become apparent from the forthcoming discussion.

Now, in order to prove the previous property, we need to establish the following theorem, which extends the analogy between the billiard and the two-body problem, and is perhaps interesting in itself.

We decompose the total momentum of the two-body system in the sum of the momentum of the centre of mass, p_G, and the relative momentum $p_{1R} = -p_{2R} = p_R$, with:

$$p_G = p_1 + p_2 \qquad p_R = \frac{m_2 p_1 - m_1 p_2}{M} \tag{8}$$
$$M = m_1 + m_2.$$

The total energy too can be split into two terms:

$$E_T = K_G + K_R = \frac{p_G^2}{2M} + \frac{Mp_R^2}{2m_1 m_2} \tag{9}$$

with, in terms of p_1 and p_2:

$$K_R = \frac{(m_2 p_1 - m_1 p_2)^2}{2Mm_1 m_2} \qquad K_G = \frac{(p_1 + p_2)^2}{2M}. \tag{10}$$

Using the rescaling (6), we can express the two last quantities terms of the components of the momentum of the equivalent billiard system, p_x and p_y:

$$K_R = \frac{1}{2M}(\sqrt{m_2}p_x - \sqrt{m_1}p_y)^2$$
$$K_G = \frac{1}{2M}(\sqrt{m_1}p_x + \sqrt{m_2}p_y)^2. \tag{11}$$

Do these quantities have some particular meaning for the billiard system? They do: in fact they represent the components of the total energy respectively normal and parallel to the hypotenuse of the triangle in figure 2.

In order to prove this, we rotate axes by an angle α. Then:

$$\begin{pmatrix} p_\parallel \\ p_\perp \end{pmatrix} = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} p_x \\ p_y \end{pmatrix}. \tag{12}$$

Recalling that $\tan\alpha = \sqrt{m_2/m_1}$, and thus $\cos\alpha = \sqrt{m_1/M}$, we get:

$$K_\perp = \frac{p_\perp^2}{2} = \frac{1}{2M}(\sqrt{m_2}p_x - \sqrt{m_1}p_y)^2$$
$$K_\parallel = \frac{p_\parallel^2}{2} = \frac{1}{2M}(\sqrt{m_1}p_x + \sqrt{m_2}p_y)^2. \tag{13}$$

Comparison between equations (11) and (13), shows that:

$$K_\perp = K_R \qquad K_\parallel = K_G. \tag{14}$$

We are now getting closer to the previously announced optical analogy.

The angle of incidence β with respect to the normal direction is defined as follows

$$\tan\beta = \frac{p_\parallel}{p_\perp} = \left(\frac{K_\parallel}{K_\perp}\right)^{1/2} = \left(\frac{K_G}{K_R}\right)^{1/2}$$
$$= \frac{\sqrt{m_1 m_2}}{m_2 p_1 - m_1 p_2}(p_1 + p_2). \tag{15}$$

There will exist a critical angle β_c that discriminates whether the particles do or do not cross each other. Such a critical angle is thus determined by the relation $K_R = V_0$, which can be written as:

$$1 + \frac{K_G}{K_R} = \frac{E_T}{V_0}$$

or

$$\tan\beta_c = \left(\frac{E_T}{V_0} - 1\right)^{1/2}. \tag{16}$$

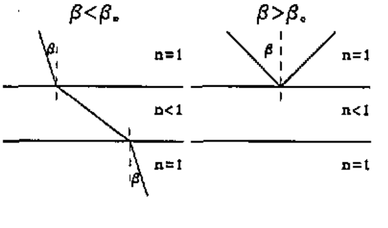
For $\beta > \beta_c$ the particles are reflected, while for $\beta < \beta_c$ they cross each other. We also see that all the physics of the collision is contained in the parameter β_c.

Now, let us define another physical system: a light beam travels in a rectangular domain, with perfectly reflecting sides; one diagonal of the rectangle is constituted of a thin slide of refracting material index $n < 1$; the rest of the rectangle is vacuum ($n = 1$). What happens when the beam of light approaches the refracting slide? Figure 3 shows a zoom of such an event: if the angle of incidence β is larger than β_c we have total reflection; if $\beta < \beta_c$ we have a double refraction after which the beam continues its propagation on a line parallel to the line of incidence. If the thickness h of the slide is small, the deflection will also be small (Born and Wolf 1980).

As a matter of fact, the optical problem is more complicated, inasmuch as the beam is always partly reflected and partly transmitted. It is at this point that the analogy ceases to be rigorous, and in the following we shall suppose that the beam is always either reflected or transmitted, according to its angle of incidence. This just means that the physics of wave propagation is more complex than the simple mechanical model, and that some phenomena have to be neglected: anyway, the analogy still keeps some heuristic value.

To conclude this section, we calculate the refractive

Figure 3. The beam is either reflected or transmitted depending on its angle of incidence. We have reflection for $\beta > \beta_c$ and transmission for $\beta < \beta_c$.



index associated to the two-body problem. From the refraction law, it follows that the relative refractive index is linked to the critical angle by the following relation:

$$n = \sin\beta_c. \tag{17}$$

Taking into account equation (16), a little algebra yields:

$$n = \left(1 - \frac{V_0}{E_T}\right)^{1/2}. \tag{18}$$

For $V_0 = 0$, $\beta_c = \pi/2$ and the particles always cross each other (no interaction); for $V_0 = E_T$, $\beta_c = 0$ and we recover the hard-core case.

Unluckily, as we had anticipated, this analogy does not work for the three-body problem. The corresponding optical system would be a cube, made of six simplexes, which makes a total number of six interfaces inside the cube. The problem is that we cannot properly define a refractive index for each interface. In fact, from equation (18), the refractive index depends on the total energy of a *couple* of particles, which, when $N > 2$, is no longer conserved. Thus, the refractive index of an interface would depend on the past history of the whole system, a situation which is hardly found in common optical materials.

4. Conclusion

The principal aim of this paper is to summarize several analogies between a few, very simple physical systems. In one case, a remarkable result was that the analogy can be demonstrated in a far more straightforward way if the problem is posed in the frame of quantum, rather than classical, mechanics.

These considerations may have some pedagogical value in showing that the difficulty of a problem closely depends on the mathematical apparatus that one displays in order to solve it. On the other hand, different physical pictures of the same mathematical problem can provide new insights and often help intuition. Since the number of physical systems about which we have an immediate intuition is very limited (and most are macroscopic, simple systems), the importance of such analogies becomes evident as soon as we want to investigate more exotic physical situations. Furthermore, it seems to us that the present day physics education (at least in Europe) is based in an exaggerated way on the manipulation of formulas, and much less on their interpretation. Developing such analogies can be, in our view, a way of acquiring familiarity with the concepts, together with a certain detachment from the mathematical tools.

A second point that may interest the physics teacher is that such systems as the N-body, one-dimensional system are both conceptually simple and easy to simulate numerically. Such very delicate concepts as ergodicity, reversibility, invariants, can be visualized in an extremely explicit way through a five-minutes numerical simulation. Finally, we point out that both the hard and soft-core two-body problems can in principle be realized experimentally, by means of a laser beam travelling in a reflecting/refracting box.

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