Stability of Bernstein–Greene–Kruskal modes

Giovanni Manfredi[a)] and Pierre Bertrand
Laboratoire de Physique des Milieux Ionisés, Université Henri Poincaré, Nancy 1, 54506 Vandoeuvre-les-Nancy, France

(Received 1 February 2000; accepted 10 March 2000)

The stability of Bernstein–Greene–Kruskal (BGK) modes is investigated in the limit of small electric potential (weak inhomogeneity). It is proven that one-hole BGK modes can be unstable, contrarily to what was observed in previous numerical simulations. A simple stability criterion is derived. In particular, it is proven that the velocity distribution must have at least three maxima for instability to occur. Numerical simulations confirm the analytical results, and extend them to the nonlinear and strongly inhomogeneous regimes. In particular, it is shown that a strong inhomogeneity has a stabilizing effect.
[S1070-664X(00)04806-0]

I. INTRODUCTION

In a seminal paper published in 1957, Bernstein, Greene and Kruskal[1] showed the existence of an infinite family of exact stationary solutions for electrostatic, collisionless plasmas described by the Vlasov–Poisson model. Such solutions (now called BGK modes) are spatially inhomogeneous, and therefore exhibit a finite self-consistent electric potential. They have continued to attract interest,[2] since they may represent the final saturated state of instabilities which are stabilized by particle trapping in the potential well formed by the growing wave.[3] Numerical results also suggest that traveling BGK waves may arise as the result of nonlinear Landau damping, a subject which is presently at the center of a stimulating debate.[4–7] Traveling BGK waves have been investigated theoretically in a series of papers,[8] particularly in the small amplitude limit. Other recent works include extensions to the two-dimensional[9] and the magnetized cases,[10] as well as applications to geophysical plasmas.[11]

In order to establish whether BGK modes can exist in an actual plasma, it is crucial to understand the stability properties of such solutions against various kinds of perturbations. Several methods have been used in the past, ranging from mode coupling analysis to thermodynamical arguments,[12] eigenvalue methods,[13] or by direct computation of the growth rates.[14] All the above techniques predict that BGK structures formed by at least two phase space vortices (''holes'') are unstable, while no rigorous result exists on the stability of one-hole structures. Numerical experiments confirm that multiple-hole BGK modes are indeed unstable, and strongly suggest that one-hole structures are stable.[15,16]

It is the scope of this paper to prove that some one-hole BGK modes can actually be unstable. In particular, it will be shown that unstable modes exist when the underlying velocity distribution has at least three maxima. Indeed, previous simulations had all considered two-stream velocity distributions, which turn out to be stable for one-hole structures.[15]

The basic idea put forward in this article is that, for weakly inhomogeneous BGK modes, the stability properties are given by the corresponding homogeneous solution, obtained by letting the electrostatic potential go to zero. This means that the growth rate varies continuously (in some sense) with the amplitude of the potential. This assumption, which is physically rather plausible, has been proven rigorously in a recent mathematical paper.[17] We will therefore investigate the stability properties of homogeneous distribution functions which are the limit of BGK modes for small electric fields, particularly in the more controversial case of one-hole structures. The conclusion is that both stable and unstable one-hole BGK modes can exist, depending on the shape of the velocity distribution.

The main result on the stability of weakly inhomogeneous BGK modes is presented in the next Section. In Sec. III, an analytical example is worked out in detail. This makes use of a special type of distribution function, composed of two cold beams and a ''water-bag'' distribution. In this case, the dispersion relation becomes an algebraic (as opposed to integral) equation, and rigorous results can be easily obtained. Section IV contains several results from numerical simulations that confirm the (linear) analytical calculations. The simulations also enable us to investigate the nonlinear saturation of unstable equilibria, as well as the case of strongly inhomogeneous BGK modes. Conclusions are presented in Sec. V.

II. GENERAL STABILITY PROPERTIES

The model considered in this paper is the one-dimensional Vlasov–Poisson system

$$\frac{\partial f}{\partial t}+v\frac{\partial f}{\partial x}+E\frac{\partial f}{\partial v}=0,$$
$$\frac{\partial E}{\partial x}=\int_{-\infty}^{\infty} f\,dv-1, \tag{1}$$

where $f(x,v,t)$ is the electron distribution function and $E(x,t)$ the electric field. In Eq. (1), and in the rest of the

a)Electronic mail: Giovanni.Manfredi@lpmi.uhp-nancy.fr

article, time is normalized to the inverse electron plasma frequency ω_{pe}^{-1}, space is normalized to the Debye length λ_D, and velocity is normalized to the electron thermal speed $V_{Te}=\lambda_D\omega_{pe}$. Ions are taken to be motionless, and their only role is to provide a uniform, neutralizing background. Periodic boundary conditions are assumed in x.

It is easy to show that any function $F(H)$ of the energy $H=v^2/2+\phi_0(x)$ (where ϕ_0 is the equilibrium electrostatic potential) is an exact stationary solution of Eq. (1). By plugging $F(H)$ into Poisson's equation, one obtains a nonlinear differential equation for the potential, which, under some condition, can have spatially periodic solutions. Such a condition can be derived easily in the weakly inhomogeneous case. By taking $\partial f/\partial t=0$ in Vlasov's equation, then dividing by v, integrating over velocity space, and finally making use of Poisson's equation, one arrives at the following equation for the equilibrium electric field $E_0=-\partial\phi_0/\partial x$

$$\frac{d^2E_0}{dx^2}+k^2(x)E_0(x)=0, \tag{2}$$

In the limit of small potentials, $k^2(x)$ becomes independent on the spatial variable and equal to

$$k_0^2=\int_{-\infty}^{\infty}\frac{1}{v}\frac{dF}{dv}dv=\int_{-\infty}^{\infty}\frac{dF}{dH}dv. \tag{3}$$

Therefore, for periodic BGK modes to exist, one needs $k_0^2>0$.[15] Notice that this condition rules out distribution functions that are monotonically decreasing with the energy $H=v^2/2$, such as the Maxwellian distribution. A velocity distribution with at least two maxima (two-stream distribution) is thus required for the existence of periodic BGK modes. The wavelength $2\pi/k_0$ represents the typical spatial period of a one-hole BGK mode.

In order to study the stability properties, one performs the usual expansion around the inhomogeneous equilibrium

$$f(x,v,t)=F(H)+f_1(x,v,t),\ \ \phi(x,t)=\phi_0(x)+\phi_1(x,t). \tag{4}$$

Inserting this expansion into Eq. (1), the linearized Vlasov equation becomes

$$\frac{\partial f_1}{\partial t}+v\frac{\partial f_1}{\partial x}+E_1\frac{\partial F(H)}{\partial v}+E_0\frac{\partial f_1}{\partial v}=0. \tag{5}$$

Now, if the equilibrium electric field E_0 is small (weakly inhomogeneous BGK), then the last term on the left-hand side of Eq. (5) is of higher order and can be neglected. Moreover, in the third term, one can make the approximation $F(H=v^2/2+\phi_0)\simeq F(v^2/2)$ (and thus neglect the inhomogeneity), since inhomogeneous corrections are of higher order. The important result is that we are left with the usual linearized Vlasov equation, in which the inhomogeneous equilibrium has disappeared. This shows that the stability properties of a weakly inhomogeneous BGK mode are entirely determined by its homogeneous limit. A mathematically rigorous proof of this theorem has been recently obtained by Guo and Strauss.[17]

By virtue of the previous result, the stability properties are therefore governed by Landau's dispersion relation

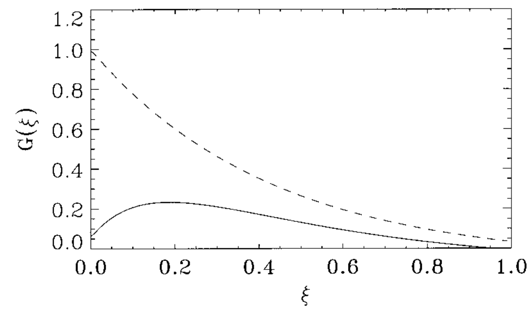

FIG. 1. Plot of the dispersion function $G(\xi)$ for an unstable three-stream plasma (solid line), and a stable two-stream plasma (dashed line), in a case for which $G'(0)\neq 0$. Only the positive ξ axis is shown. The plots correspond to the velocity distribution of Eq. (24) with $\beta=0.045$, $T=0.05$ for the unstable case (solid line), and $\beta=0.0$, $T=0.05$ for the stable one (dashed line).

$$k^2=\int_{-\infty}^{\infty}\frac{dF/dv}{v-z}dv\equiv G(z), \tag{6}$$

where $z=\omega/k$, and $G(z)$ is defined by the last equality in Eq. (6). For $z=0$, we obtain k_0^2, the wave number of weakly inhomogeneous BGK modes, given by Eq. (3). Indeed, it is not surprising that, in the limit of zero field, BGK modes tend to a homogeneous state for which the dispersion relation has at least the solution $\omega=0$, i.e., a marginally stable solution. However, other solutions with $\mathrm{Im}\,\omega>0$ might exist, corresponding to an instability.

We first notice that BGK modes, being steady-state solutions, must tend to linear waves with $\mathrm{Re}\,\omega=0$. Therefore, in order to investigate stability, we shall consider the branch of the dispersion relation, Eq. (6), corresponding to $\mathrm{Re}\,z=0$, and write $z=\mathrm{i}\xi$, with ξ real. One obtains

$$G(\xi)=\int_{-\infty}^{\infty}\frac{v}{v^2+\xi^2}\frac{dF}{dv}dv, \tag{7}$$

where we have used the fact that $F(v)$ is an even function. The origin $\xi=0$ is a solution of the dispersion relation $k^2=G(\xi)$ corresponding to k_0. The main point is that, if $G(\xi)$ increases for small positive values of ξ, then another solution with $\xi>0$ will necessarily exist for the same wave number k_0 (Figs. 1 and 2). This is true only if $G(\xi)$ goes to zero for

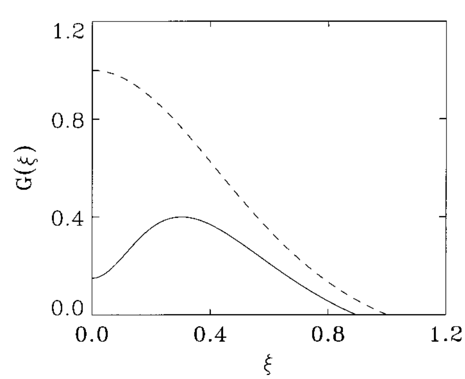

FIG. 2. Plot of the dispersion function $G(\xi)$ for a ''beam-water-bag'' distribution, as given by Eq. (19). In this case, $G'(0)=0$. The parameters used are $u=1$, $a=0.25$, $\beta=0.05$ (solid line, unstable), and $u=1$, $a=0.25$, $\beta=0$ (dashed line, stable).

some larger value of ξ, which will also be proven in the following.

Since we are interested in the behavior near $\xi=0$, we expand $G(\xi)$ in a power series

$$G(\xi)=G(0)+\xi G'(0)+\frac{\xi^2}{2}G''(0)+\cdots, \tag{8}$$

where the apex stands for derivative with respect to ξ. Obviously $G(0)=k_0^2$. Let us evaluate the first derivative of $G(\xi)$

$$G'(\xi)=-\xi\int_{-\infty}^{\infty}\frac{1}{v^2+\xi^2}\frac{d^2F}{dv^2}dv, \tag{9}$$

where we have integrated by parts. In order to evaluate $G'(\xi=0)$, we add and subtract to the right-hand side of Eq. (9) terms proportional to

$$\int_{-\infty}^{\infty}\frac{dv}{v^2+\xi^2}=\frac{\pi}{|\xi|}. \tag{10}$$

One obtains

$$\begin{aligned}G'(0)&=\lim_{\xi\to 0}-\xi\left(\int_{-\infty}^{\infty}\frac{F''(v)-F''(0)}{v^2+\xi^2}dv\right.\\&\qquad\left.+F''(0)\int_{-\infty}^{\infty}\frac{dv}{v^2+\xi^2}\right)\\&=-F''(0)\pi\,\mathrm{sgn}(\xi).\end{aligned} \tag{11}$$

The symbol $\mathrm{sgn}(\xi)$ on the right hand-side of Eq. (11) means that $G'(0)$ has a different sign depending on whether the limit $\xi\to 0$ is approached for positive or negative values of ξ. Note that we have written $F''(v)$ for d^2F/dv^2. If $G'(0)\neq 0$ the expansion needs not be carried out at higher orders, and the dispersion relation becomes

$$k^2=G(\xi)=k_0^2-|\xi|\pi F''(0)+\cdots. \tag{12}$$

Therefore, when $F''(0)<0$, $G(\xi)$ is an increasing function of its argument for small positive values of ξ. Furthermore, we shall prove later on that there exists a value $\xi^*>0$, for which $G(\xi^*)=0$. We can conclude that a growing solution ($\xi>0$) must necessarily exist for the wave number k_0 (Fig. 1). It was established earlier [Eq. (3)] that, for periodic BGK modes to exist, the velocity distribution must possess at least two maxima. Now, the instability condition requires that a further maximum be present at $v=0$ [because, since $F(v)$ is an even function, $F'(0)=0$]. Therefore, a three-stream distribution is needed to guarantee the existence and instability of a one-hole BGK mode.

In the opposite case ($F''(0)>0$), nothing definite can be said for the wave number k_0. However, it is obvious that a growing solution always exists for $k=k_0/N$, with $N\geqslant 2$ (Fig. 1). This is the standard result that multiple-hole BGK modes are unstable. Two-stream distribution functions, usually considered in the past, belong to this class.

Finally, when the distribution function is flat at $v=0$ ($F''(0)=0$), the expansion must be carried out to second order. The second derivative of $G(\xi)$ at $\xi=0$ is

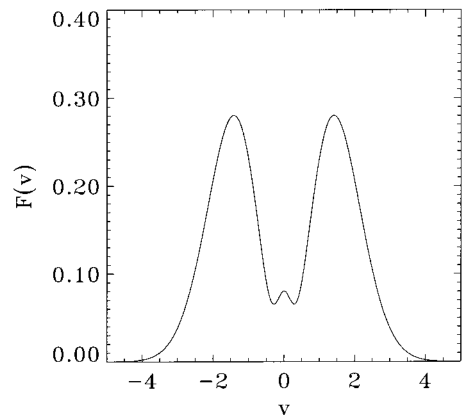

FIG. 3. Velocity distribution for an unstable three-stream plasma, as given by Eq. (24) with $\beta=0.045$, $T=0.05$.

$$G''(0)=\lim_{\xi\to 0}\frac{G'(\xi)-G'(0)}{\xi}=\int_{-\infty}^{\infty}\frac{F''(0)-F''(v)}{v^2}dv, \tag{13}$$

where we have used Eqs. (9)–(11). The dispersion relation becomes in this case

$$k^2=G(\xi)=k_0^2+\tfrac{1}{2}\xi^2G''(0)+\cdots. \tag{14}$$

In order for $G(\xi)$ to be an increasing function for small positive ξ, one must have $G''(0)>0$. This is our second result on the instability of one-hole BGK modes. The qualitative behavior of $G(\xi)$ is shown on Figs. 1 and 2, both for a stable two-stream and an unstable three-stream velocity distribution. Fig. 1 refers to a case for which $G'(0)\neq 0$, whereas Fig. 2 to a case for which $G'(0)=0$. An example of an unstable distribution is shown on Fig. 3.

The criteria derived in the previous paragraphs are *sufficient* conditions for the instability of one-hole BGK modes. Notice that they cover all possibilities, since $G''(0)$ is never zero when $F''(0)=0$. Therefore, the first two terms of the expansion given by Eq. (8) completely determine the relevant stability properties.

To complete the proof, we need to show that there exists a value $\xi^*>0$, for which $G(\xi^*)=0$. We do this by proving that, for large enough values of ξ, $G(\xi)$ becomes negative. Let us change variable to $u=1/\xi$ in Eq. (7), and define $\Gamma(u)=G(1/\xi)$. One obtains

$$\Gamma(u)=u^2\int_{-\infty}^{\infty}\frac{v}{u^2v^2+1}\frac{dF}{dv}dv. \tag{15}$$

We want to expand $\Gamma(u)$ near $u=0$. One obtains immediately that $\Gamma(0)=0$. Evaluating the derivatives of $\Gamma(u)$, it is found that $d\Gamma/du|_{u=0}=0$, $d^2\Gamma/du^2|_{u=0}=-2$. Therefore, the expansion around $u=0$ is $\Gamma(u)=-u^2+O(u^4)$. Going back to the function $G(\xi)$, one finally obtains the following expansion, valid for $\xi\gg 1$

$$G(\xi)=-\xi^{-2}+O(\xi^{-4}). \tag{16}$$

Since $G(0)>0$, $G(\xi)$ must change sign at some finite $\xi^*>0$. This completes the instability proof. Notice that the last result simply means that, for small wave numbers, the growth rate $\gamma=\mathrm{Im}\,\omega$ is proportional to the wave number, i.e., $\gamma \simeq \xi^* k$. This is the expected result for fluid models, and it is well-known that the limit $k \ll 1$ corresponds to the fluid limit.

So far, we have proven that the homogeneous limit of a one-hole BGK mode can be unstable if some criteria are satisfied. In virtue of the previous discussion [Eq. (5)], these stability properties can be extended to weakly inhomogeneous BGK modes, a result that has been proven rigorously by Guo and Strauss,[17] both for linear and nonlinear instability. This completes the proof that some one-hole BGK modes (namely those with a three-stream velocity distribution) are indeed unstable.

III. AN ANALYTICAL EXAMPLE

In order to illustrate the previous results, it is useful to consider a special equilibrium velocity distribution, for which most calculations can be performed analytically. Let us take

$$F(v) = \frac{1-\beta}{2}[\delta(v-u)+\delta(v+u)] + \beta W_a(v), \tag{17}$$

where $\delta(v)$ is the Dirac delta, and $W_a(v)$ is the so-called ''water-bag'' function, which is constant and equal to $(2a)^{-1}$ for $v<|a|$ and zero elsewhere. The distribution of Eq. (17) is made of two cold streams at velocities $\pm u$, plus a third ''warm'' stream centered at $v=0$, which is modeled by the water-bag function. The two-stream case is recovered for $\beta=0$.

The dispersion relation is obtained by inserting Eq. (17) into Eq. (6). One finds

$$k^2 = G(z) = \frac{1-\beta}{2}\left[\frac{1}{(z-u)^2}+\frac{1}{(z+u)^2}\right] + \frac{\beta}{z^2-a^2}, \tag{18}$$

where $z=\omega/k$. If we now consider the case of a purely imaginary z, and define $\xi = \mathrm{Im} z = \gamma/k$, we obtain

$$k^2 = G(\xi) = \frac{1-\beta}{2}\,\frac{u^2-\xi^2}{(\xi^2+u^2)^2} - \frac{\beta}{\xi^2+a^2}. \tag{19}$$

The plot of $G(\xi)$ is shown on Fig. 2 for two sets of values of the parameters a, u, and β. These correspond to either a stable or an unstable plasma.

The characteristic wave number of a one-hole BGK mode is (in the homogeneous limit)

$$k_0^2 = G(0) = \frac{1-\beta}{u^2} - \frac{\beta}{a^2}, \tag{20}$$

and one must have $k_0^2>0$. The stability properties are determined by the behavior of the dispersion relation, Eq. (19), for small values of ξ. Expanding $G(\xi)$ for $\xi \ll a < u$, one obtains

$$G(\xi) \simeq G(0) + G''(0)\frac{\xi^2}{2} = k_0^2 + \left(\frac{2\beta}{a^4} - 6\frac{1-\beta}{u^4}\right)\frac{\xi^2}{2}. \tag{21}$$

It is easy to verify that the expressions for $G(0)$ and $G''(0)$ could have been obtained by inserting the velocity distribution of Eq. (17) respectively into Eqs. (3) and (13), in agreement with the general theory presented in Sec. II. Note that, since the velocity distribution is flat at $v=0$, the first order term is absent in the expansion for $G(\xi)$. Combining the condition for existence of periodic BGK modes ($k_0^2>0$) and the condition for instability [$G''(0)>0$], one obtains the system

$$\begin{aligned} &\frac{1-\beta}{u^2} - \frac{\beta}{a^2} > 0, \\ &\frac{\beta}{a^4} - 3\frac{1-\beta}{u^4} > 0. \end{aligned} \tag{22}$$

Therefore, β must satisfy the following inequalities

$$\frac{3a^4}{u^4+3a^4} < \beta < \frac{a^2}{u^2+a^2}, \tag{23}$$

which imply $a<\sqrt{3}u$ and $\beta<0.25$. This means that, in order to have instability, the ''temperature'' of the central water-

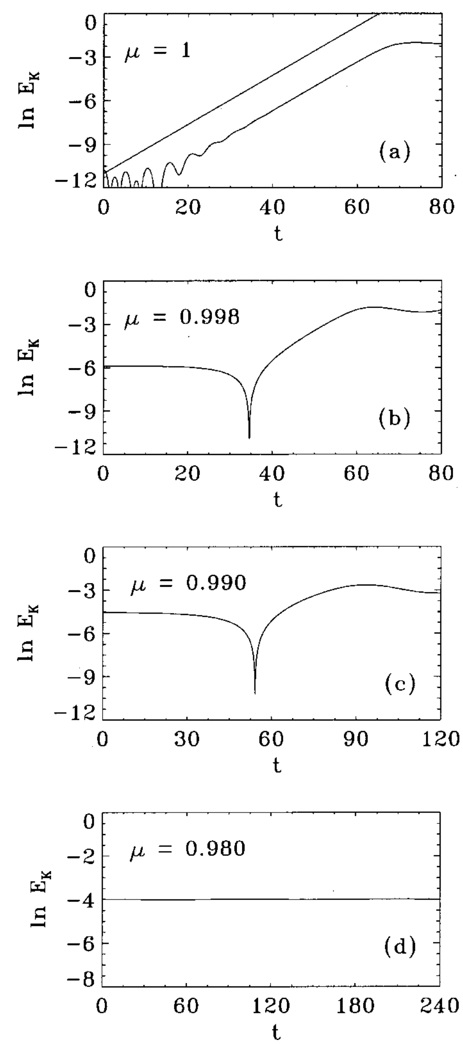

FIG. 4. Time evolution of the fundamental mode of the electric field for $\beta=0.045$, $T=0.05$. The inhomogeneity parameter is $\mu=1$ (a) (homogeneous case); $\mu=0.998$ (b); $\mu=0.99$ (c); $\mu=0.98$ (d). The straight line in (a) corresponds to the exact linear result for the homogeneous case, $\gamma=0.169$.

bag (which is proportional to a^2) must not exceed a certain value; at the same time, the fraction β of particles in the water-bag distribution must be relatively small. Figure 2 shows the function $G(\xi)$ for $u=1$ and $a=0.25$. From Eq. (23), instability occurs when $0.0116<\beta<0.0588$. The curves plotted on Fig. 2 correspond to $\beta=0.05$ (unstable) and $\beta=0$ (stable).

IV. A NUMERICAL EXAMPLE

As a further example of a distribution function satisfying the criteria for instability, let us take the three-stream distribution

$$F(H)=\frac{1}{\sqrt{2\pi}}\left(2(1-\beta)H\exp(-H)+\frac{\beta}{\sqrt{T}}\exp(-H/T)\right), \tag{24}$$

where $H=v^2/2+\phi_0(x)$, and $0\leqslant\beta\leqslant1$. The two-stream distribution is recovered for $\beta=0$. From Eq. (3), one obtains the wave number of one-hole structures in the zero field limit, $k_0^2=(1-\beta)-\beta/T$. The condition for the existence of periodic BGK modes is $k_0^2>0$. This becomes, for the above distribution, $\beta<T/(1+T)$. The relevant criterion for instability is $d^2F/dv^2|_{v=0}<0$. This requires that $\beta>2T^{3/2}/(1+2T^{3/2})$. Finally, we have unstable solutions if β and T satisfy the following condition

$$\frac{2T^{3/2}}{1+2T^{3/2}}<\beta<\frac{T}{1+T}. \tag{25}$$

It is easy to prove that the above inequalities imply $T<0.25$, and $\beta<0.2$. The parameter β represents the fraction of particles situated in the central stream (around $v=0$). Therefore, an unstable distribution appears to possess two dominant streams at finite opposite velocities, plus a small central ''bump'' situated in the hole in between the two streams (Fig. 3).

Using Eq. (24), Poisson's equation for the equilibrium electrostatic potential ϕ_0 becomes

$$\frac{d^2\phi_0}{dx^2}=1-\mu[(1-\beta)(1+2\phi_0)\exp(-\phi_0)+\beta\exp(-\phi_0/T)], \tag{26}$$

where μ is a parameter quantifying the deviation from the homogeneous equilibrium: $\mu=1$ for the homogeneous limit of a BGK mode, and $0<\mu<1$ for a truly inhomogeneous BGK equilibrium.

Several numerical simulations have been performed with a Vlasov Eulerian code based on a flux balance technique.[18] Here we present the results from a set of simulations using the distribution function of Eq. (24), with $T=0.05$ and $\beta=0.045$. According to the theory presented in the previous Sections, weakly inhomogeneous one-hole BGK modes cor-

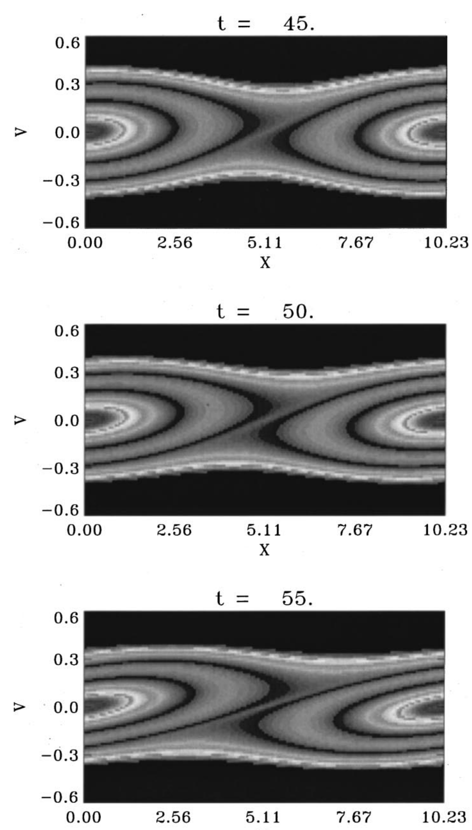

FIG. 5. Phase space portrait of the distribution function at various times between $t=45$ and $t=60$, for the case $\beta=0.045$, $T=0.05$, and $\mu=0.99$ [Eq. (24)]. Regions for which $f<0.062$ are white, whereas regions for which $f>0.080$ are black.

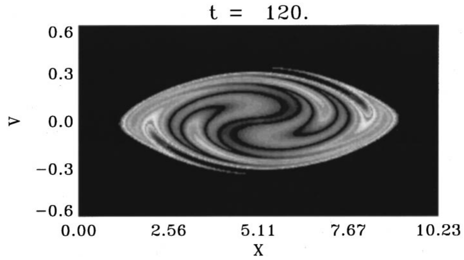

FIG. 6. Same as Fig. 5, at the end of the simulation ($t=120$).

responding to such parameters should be unstable [see Eq. (25)]. The simulations were performed on a phase space grid $N_X\times N_V=300\times 1000$, and time step $\Delta t=0.1$. The equilibrium distribution was perturbed on the fundamental wave number, with a small amplitude $\epsilon\simeq 10^{-5}$. Note that the fundamental wave number k_F of a one-hole BGK mode varies with the inhomogeneity, and is therefore, a function of the parameter μ. For the homogeneous case ($\mu=1$), this can be computed analytically from Eq. (3), and yields $k_F=k_0=(1-\beta-\beta/T)^{1/2}\simeq 0.2345$. For the inhomogeneous case, k_F must be computed numerically. One obtains, for instance, $k_F\simeq 0.4558$ for $\mu=0.998$, and $k_F\simeq 0.6101$ for $\mu=0.990$.

In Fig. 4, we present the time evolution of the fundamental mode of the electric field, for various values of the inhomogeneity parameter. In the homogeneous case ($\mu=1$) the growth rate γ can be computed analytically from the dispersion relation, Eq. (7). One obtains $\gamma\simeq 0.169$, a value that closely matches the result of the simulation [Fig. 4(a)]. For truly inhomogeneous BGK modes ($\mu<1$), it is not possible to compute the growth rate analytically. However, it is clear that the instability persists for $\mu=0.998$ and 0.99 [Figs. 4(b) and 4(c)]. More surprisingly, we have found that, when $\mu=0.98$ or smaller, the system becomes stable [Fig. 4(d)]. In other words, a one-hole BGK mode which is unstable for a weak inhomogeneity, becomes stable when the inhomogeneity is strong enough. Note that this result cannot be deduced from the theory presented in Sec. II, which is only valid for quasi-homogeneous equilibria.

Figure 5 shows the phase space portrait of the distribution function for the case $\mu=0.99$. Between $t=0$ and $t=45$ there is little visible evolution. Subsequently, the two halves composing the single hole of this BGK mode start attracting each other and merge. Over longer times ($t=120$, Fig. 6), a new equilibrium appears, which is still an (approximate) one-hole BGK mode. Note, however, that the ''center-of-mass'' of the distribution function has undergone a shift of half the fundamental wavelength $2\pi/k_F$. This is also visible in the plot of the electric field (Fig. 7): At $t\simeq 55$ the electric field goes to zero, and then grows again, but with a different sign, corresponding to a phase shift of 180 degrees. For longer times, the electric field is actually larger than for the initial equilibrium. This phase shift corresponds to the sudden drop of the amplitude of the first mode of the electric field, visible in Fig. 4(c).

The (spatially averaged) velocity distribution is shown on Fig. 8 for the case $\mu=0.998$. We show this case because, for smaller values of μ, the central bump in the velocity distribution is very small and hardly visible. Indeed, as was proven earlier on, the relative weight of the central stream, compared to the two streams at nonzero velocity, is proportional to the parameter β, and $\beta<0.2$. Furthermore, the small central bump tends to be smoothed out by the density modulation (inhomogeneity).

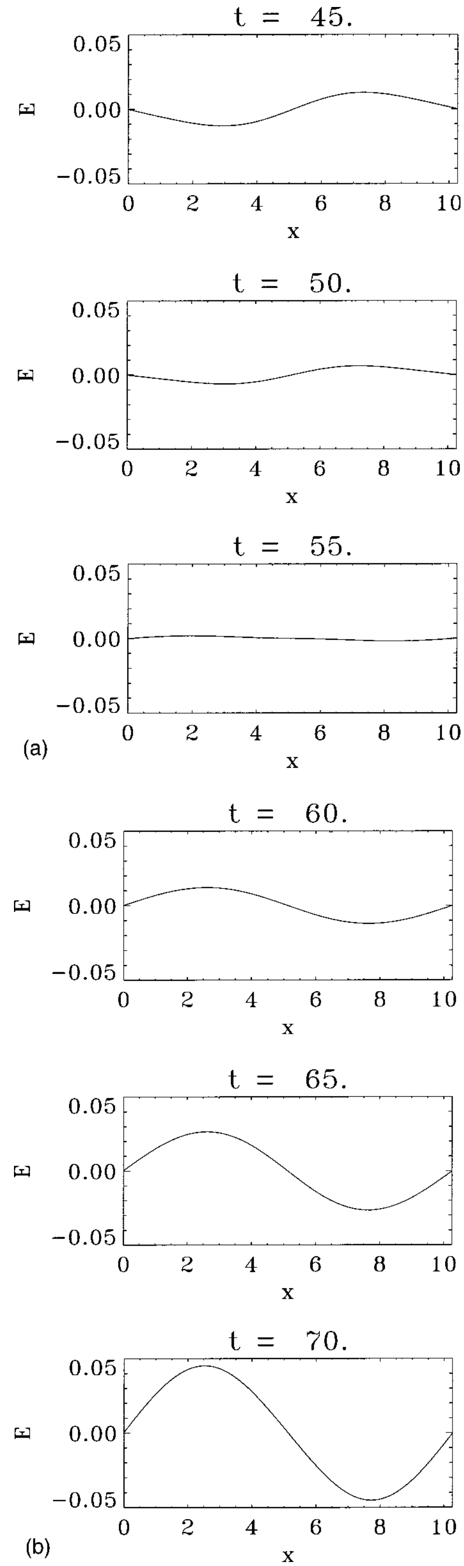

FIG. 7. Plot of the electric field at different times, for the case $\beta=0.045$, $T=0.05$ and $\mu=0.99$. Same run as Fig. 5.

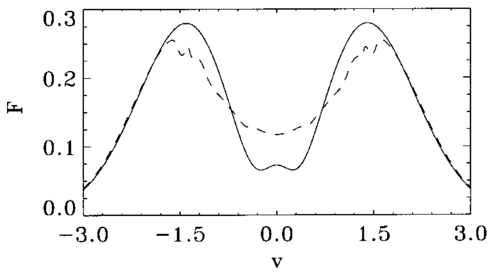

FIG. 8. Plot of the spatially averaged velocity distribution for the case $\beta = 0.045, T = 0.05$, and $\mu = 0.998$, at times $t = 0$ (solid line) and $t = 80$ (dashed line).

We observe from Fig. 8 that the final equilibrium ($t = 80$) presents no bump in the central part of the velocity distribution. From this and previous results, we can conclude that the system evolves from an unstable one-hole equilibrium towards another one-hole equilibrium, which appears to be stable.

V. CONCLUSION

In this paper, we have shown that some one-hole BGK modes can be unstable under certain conditions. A rigorous (linear) theory has been developed, which is valid for weakly inhomogeneous BGK modes.

On the basis of previous numerical simulations, one-hole BGK modes were generally thought to be stable. The reason for this misconception is that only velocity distributions composed of two streams were considered. For these, we have proven that the corresponding one-hole BGK mode is always marginally stable, i.e., it is a mode with vanishing growth rate. However, for velocity distributions composed of at least *three* streams, instability can occur when some conditions are satisfied. These conditions were derived explicitly for a generic velocity distribution.

As an analytical example to illustrate this result, we have used a velocity distribution made of two cold counterstreaming beams, plus a water-bag distribution centered at zero velocity. This is supposed to mimic, in a simplified way, a general three-stream distribution. The advantage of such a beam-water-bag distribution is that it enables one to compute explicitly the dispersion relation, which becomes an algebraic, rather than integro-differential, expression. The subsequent stability analysis is in agreement with the general theory presented in Sec. II.

The question of the stability of strongly inhomogeneous modes cannot be investigated analytically, since the arguments employed here are not valid far from the homogeneous limit. In order to address this point, we have performed several numerical simulations with a Vlasov code. The main result is rather surprising: One-hole modes that are unstable in the homogeneous limit, become stable when the inhomogeneity is stronger than a certain threshold. Another question that we were able to address via computer experiments is the nonlinear saturation of unstable BGK modes. It was shown that an unstable one-hole BGK structure evolves towards another (stable) one-hole structure by modifying the shape of the velocity distribution.

In conclusion, we have presented detailed analytical and numerical results proving that some one-hole BGK modes, which were previously thought to be stable, can actually be unstable. A consequence of this result is that it restricts the class of BGK modes which may represent the final saturated state of linear instabilities. Furthermore, given the mathematical analogy between the one-dimensional Vlasov–Poisson system and the two-dimensional Euler equation,[4] these results may also be relevant to fluid problems, such as the Kelvin–Helmoltz instability.

ACKNOWLEDGMENTS

It is a pleasure to thank Marc Feix for many years of stimulating discussions on this subject. We also thank Fernando Haas for a useful comment.

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